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STATISTICAL STUDY OF MICROWAVE PROPAGATION THROUGH THE TURBULENT ATMOSPHERE

(Determination of the Equivalent Scattering Cross Section and Amplitude and Phase Fluctuations)

Dr. J. Nikolis
Asst. Lecturer of the Polytechnic Institute of Greece; Supervisor, Radio-Electrical Research Laboratory of the Hellenic Telecommunication Organization

Translation of "Statistike spouge tes diadoseos ton mikrokumaton dia strobilodous atmosphairas (kathorismos tes isodunamou dia tomes skedaseos kai ton diakumanseon eurous kai phaseos)," Technika Chronika (Greece), vol. 41, No. 2A, pp. 1-26, April 1964.

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ABSTRACT

This article is intended to provide researchers spe- /1* cializing in scientific radio electronics, with an exact theoretical up-to-date analysis on the very essential (especially for us) subject of the study of the microwave propagation conditions through the troposphere, especially in the case in which transmitter (source) and receiver (point of observation) are in line of sight contact. In the following development mainly the latest works of Soviet scientists were taken into consideration (L. A. Chernov, V. A. Krasilnikov, V. I. Tatarski, A. M. Obukhov, A. N. Kolmogorov, and L. D. Landau) through which the rather exact and mathematically disciplined statistical description of the propagation medium was achieved. Many original related works, determining similar parameters and attributed to the first who dealt with the problem (H. G. Booker, W. E, Gordon, K. Bullington, C. L. Pekeris, A. Wheelon, H. Staras, F. Villars, V. F. Weisskoopf, E. G. S. Megaw, R. A. Silverman, etc.) result as special cases (by special choice of the turbulence spectrum in each case) of the general theoretical concepts developed below.

Introduction

The problem of electromagnetic wave scattering by the lower layers of the atmosphere owes its development basically to the fact that the phenomenon uncovered the possibility of radioelectric connection, on the range of

*Numbers given in the margin indicate the pagination in the original foreign text.

approximately 50 - 10 000 $\frac{Mc}{s}$, of points situated beyond the optical horizon.

However, the concept of "optical horizon" (except in some cases of very long or very short coupling) must be studied statistically: depending upon the instantaneous value of the equivalent slope* of the refraction index in the region of coupling, the conformally transformed earth radius is pos-sometimes able to allow line of sight contact of the two points to be joined and sometimes not. Thus it is possible (and this is only experimentally proven) for a radioelectric coupling of medium length to be considered for a certain percentage of time as "tropospheric beyond the horizon" and for the rest of the study time as coupling "of long line of sight contact". Either way, the nature of the problem does not change: in small distances the primary field to the receiver is, depending on the occasion, the one that corresponds to the level of free space, or that corresponding to the reduction due to refraction over the surface of the Earth. In the first case the field reduces proportionally to the square of the distance L between transmitter and receiver, in the second, approximately exponentially to it.

On the other hand the field due to scattering** which for small distances superimposed on the main component gives the fluctuation about the mean value, which the main component determines, decreases mainly linearly*** with the distance L for line of sight connection, and proportional to L^n for couplings beyond the horizon. It is, therefore, obvious that as long as the length of coupling gradually increases, there develops for each of the above cases a limit L_{cr} (faster for the beyond the horizon case and much later for the case of the line of sight connection) for which the levels of the main components for each case contribute to the formation of the instantaneous value of the field of reception under the same or relative range of magnitude to the component due to scattering. In the region of this limit the experimentally proven, statistical morphology of the field amplitude and phase fluctuation, being directly dependent for a given frequency upon the dimension of the turbulence scale, gives valuable information on the "propagation mechanism" for each case.

* See reference P. Misme "Le Gradient Equivalent" Annales des telecommunications, vol. 15, Nos. 3-4, March-April 1960.

Also J. Nikolis "On the conditions of propagation of electromagnetic radiation through the troposphere. Correlation of radioelectrical and radiometeorological papameters", these for doctorate, June 1962.

** It is understood to mean scattering in the troposphere. The problem of electromagnetic radiation scattering by a rough reflective surface (sea, ground) will be dealt with in our forthcoming article containing special experimental work.

***See below.

PART A.

DETERMINATION OF THE EQUIVALENT SCATTERING CROSS SECTION FOR THE ELECTROMAGNETIC RADIATION PROPAGATION THROUGH TURBULENT TROPOSPHERE

The problem of electromagnetic wave scattering through nonhomogeneous /2 and nonisotropic atmosphere is stated as follows: a plane monochromatic electromagnetic wave is incident on an atmospheric volume V; due to the existing distributions* of the refraction index inside this volume resulting from turbulence, the oncoming wave is being scattered (i.e. being diffracted).

It is required to find the equivalent surface of scattering** (or the mean density of the scattered radiation) in a given direction ϕ, θ.

We will assume that the distribution of the refraction index $n(\bar{r})$ inside the volume V is statistically "random" and independent of time.

A. Assume that the conductivity of the troposphere is zero and its (relative) magnetic permeability is unity.

The Maxwell equations under these conditions are written:

$$rot\,\overline{E} = jk\overline{H} \qquad (1)$$

$$rot\,\overline{H} = -jk\varepsilon\overline{E} \qquad (2)$$

$$div\,\varepsilon\overline{E} = 0 \qquad (3)$$

where $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$, s, the relative dialectric constant and E, H the amplitude of the electric and magnetic field, respectively..

Taking rot = Cure of both sides of equation (1) above and substituting from (2) we get:

$$-\nabla^2\overline{E} + grad\,.\,div\overline{E} = k^2.\,\varepsilon\,\overline{E}.$$

since $$div\,\varepsilon\overline{E} = \varepsilon\,.\,div\,.\,\overline{E} + \overline{E}\,.\,grad\,\varepsilon = 0$$

we have $$div\overline{E} = -\overline{E}\,grad\,.\,log\,\varepsilon$$

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* See below

** Meaning the per unit of solid angle, unit of scattering volume and unit of incoming power density, power of reception.

and since $s = n^2$ (r) we finally get:

$$\nabla^2 E + k^2 . n^2 . \overline{E} + 2grad\,(\overline{E} . grad . \log n) = 0 \qquad (4)$$

We assume that the fluctuations of the refraction index n are small, i.e. $| n_i - \bar{n} | << 1$. If we call n_1 the deviation of n from the mean value: $n = \bar{n} + n_1$, since $\bar{n} \cong 1$*, replacing the n in Eq. (4) by $1 + n_1$ we get:

$$\nabla^2 \overline{E} + k^2 . \overline{E} = - 2grad\,(\overline{E} . grad . \log\,[1+n_1]\,) - \\ - 2 . k^2 . n_1 . \overline{E} - k^2 . n_1^2 \overline{E} \qquad (5)$$

To solve Eq. (5) we put:

$$\overline{E} = \overline{E}_0 + \overline{E}_1 + \overline{E}_2 + \; . \; . \; . \; . \qquad (5)'$$

where the n^{th} term has the order of magnitude n_1^n (Born's approximation). Substituting in Eq. (5) and equating to zero each group of terms of the same order of magnitude we obtain:

$$\nabla^2 \overline{E}_0 + k^2 \overline{E}_0 = 0 \qquad (6)$$

$$\nabla^2 \overline{E}_1 \; k^2 \; \overline{E}_1 = - \; 2k^2 n_1 \overline{E}_0 - 2grad\,(\overline{E}_0\, grad\, n_1) \qquad (7)$$

In Eq. (7) the quantity log $(1 + n_1)$ which appears in Eq. (5) has been expanded in the powers of n_1, i.e.

$$\log\,(1+n_1) - n_1 - \frac{n_1^2}{2} + \cdot \cdot \cdot \cdot$$

The quantity E_o represents the amplitude of the field strength of the oncoming wave which, since it was taken as plane is $E_o = A_o \cdot e^{j\bar{k}\bar{r}}$. The quantity $\overline{E}_1$ represents the amplitude of the scattered wave (the terms $\overline{E}_2$ ...... are omitted due to their infinitesimal size).

* The value n-1 near the ground is of the order of magnitude $\sim 3.10^{-4}$.

The solution now of the equation $\nabla^2\psi(\bar{r}) + k^2 \cdot \psi(\bar{r}) = f(\bar{r})$ (which corresponds to the outgoing waves) is of the form

$$\psi(\bar{r}) = -\frac{1}{4\pi} \int_V f(\bar{r}') \cdot \frac{e^{jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \, dV' \qquad (8)$$

where $\bar{r}'$ is the variable vector which originates at the zero of the coordinate system which we place inside the volume of diffusion - in different points of this volume.

If the point of observation $\bar{r}$ is located at a great distance from the volume V as compared to the dimensions of this volume then for all the values $\bar{r}'$ the quantity $|\bar{r} - \bar{r}'|$ is approximately constant and equal to $r = |\bar{r}|$.

Under these conditions the quantity $|r - r'|$ can be expanded in a power series in the ratio $\frac{r}{r}$, i.e.:

$$|\bar{r}-\bar{r}'| = \bar{r} - \bar{\delta}_0 \bar{r}' + \frac{1}{2r}\left[r'^2 - (\bar{\delta}_0 \bar{r}')^2\right] + \cdots \quad \text{where} \quad \bar{\delta}_0 = \frac{\bar{r}}{r}$$

the unit vector drawn from the origin of the coordinate system (inside the volume V) towards the point of observation. If the relation:

$$\frac{k}{2r}\left[r'^2 - (\bar{\delta}_0 \cdot \bar{r}')^2\right] << 1 \qquad (8)'$$

is valid for all the values of r', i.e. if the dimensions L of the scattering volume satisfy the condition $\lambda . r >> L^2$, then

$$e^{jk|\bar{r}-\bar{r}'|} \backsim e^{jk(r-\bar{\delta}_0 \bar{r}')} \qquad (8)''$$

Further, substituting in the denominator of equation (8) $|\bar{r} - \bar{r}'| \sim r$ we get:

$$\psi(\bar{r}) - \frac{1}{4\pi} \cdot \frac{e^{jkr}}{r} \cdot \int_V f(\bar{r}') \cdot e^{-jk\bar{\delta}_0 \bar{r}'} \cdot dV' \qquad (9)$$

Using now relation (9) to solve (7) we get

$$\bar{E}_1(\bar{r}) = \frac{k^2}{2\pi} \cdot \frac{e^{jkr}}{r} \cdot \int_V n_1(\bar{r}') \cdot \bar{A}_0 \cdot e^{j \cdot (\bar{k} \cdot \bar{r}' - k \cdot \bar{\delta}_0 \bar{r}')} dV' +$$

$$+ \frac{1}{2\pi} \cdot \frac{e^{jkr}}{r} \cdot \int_V grad\,(e^{j\bar{k}\cdot\bar{r}'} \bar{A}_0 grad n_1(\bar{r}')) \cdot e^{-jk \cdot \bar{\delta}_0 \cdot \bar{r}'} dV' \quad (10)$$ /3

The second integral of (10) is being transformed by making use of Gauss' theorem:

$$\int_V u\, grad\varphi . dV' = \int_s \varphi . u\, d\sigma - \int_V \varphi . grad u . dV'$$

where s is the surface surrounding the volume V and dσ an element of this surface.

The surface integral is equal to zero as long as the surface of integration can be extended beyond the limits of the volume V. Since:

$$grad\ .\ e^{-jk\bar{\delta}_0 \bar{r}} = -jk . \bar{\delta}_0 e^{-jk . \bar{\delta}_0 . \bar{r}} .$$

we have:

$$\int_V grad\,(e^{-j\bar{k}.\bar{r}'} \bar{A}_0\, grad n_1(\bar{r}')) . e^{-jk\bar{\delta}_0 \bar{r}'} dV' =$$

$$= jk\bar{\delta}_0 . \bar{A}_0 . \int_V \left(grad n_1(\bar{r}') \right) . e^{j(\bar{k}\bar{r}' - k\bar{\delta}_0 \bar{r}')} . dV'$$

then:

$$E_1(\bar{r}) = \frac{k^2 . e^{jkr}}{2\pi . r} . \bar{A}_0 . \int_V n_1(\bar{r}') . e^{j \cdot (\bar{k} - k . \bar{\delta}_0)\bar{r}'} . dV' +$$

$$+ \frac{jke^{jkr}}{2\pi r} . \bar{\delta}_0 . \int_V \bar{A}_0 grad n_1(\bar{r}') . e^{j(\bar{k} - k\bar{\delta}_0)\bar{r}'} dV'$$

or

$$\boxed{E_1(\bar{r}) = \frac{k^2 \cdot e^{jkr}}{2\pi r} \cdot c_1 \cdot \bar{A}_0 + \frac{j\,k \cdot e^{jkr}}{2\pi r}\, c_2 \cdot \bar{\delta}_0} \qquad (11)$$

where:

$$c_1 = \int_V n_1(\bar{r}') \cdot e^{j(\bar{k} - k\bar{\delta}_0)\bar{r}'}\, dV' \qquad (11)'$$

and

$$c_2 = \bar{A}_0 \cdot \int_V grad\, n_1(\bar{r}') \cdot e^{j(\bar{k} - k\bar{\delta}_0)\bar{r}'}\, dV \qquad (11)''$$

Both the terms of (11) represent spherical waves the amplitude and phase of which depend on the fluctuation of the refraction index $n_1(\bar{r}')$, inside the volume V. The second term describes a longitudinal electrical field. If we transform the expression for c_2, by making use of Gauss' theorem, we get:

$$c_2 = -\bar{A}_0 \int_V n_1(\bar{r}') \cdot grad\left[e^{j(\bar{k} - k\bar{\delta}_0)\bar{r}'} \right] dV$$

but:

$$grad\; e^{-j(\bar{k} - k\bar{\delta}_0)\bar{r}'} = -j(\bar{k} - k\bar{\delta}_0)\, e^{-j(\bar{k} - k\bar{\delta}_0)\bar{r}'}$$

so:

$$c_2\,\bar{\delta}_0 = +j(k\,\bar{\delta}_0 - k)\,\bar{A}_0 \int_V n_1(\bar{r}') \cdot e^{j(\bar{k} - k\bar{\delta}_0)\bar{r}'}\, dV' = $$

$$= jk \cdot c_1\,\bar{A}_0$$

Therefore, the scattered wave is purely transversal.

B. Calculate the flux density of the scattered energy.

The indicated value of the energy density during one period is

$$\bar{S} = \frac{c}{8\pi} \cdot \mathrm{Re}\,[E_1\; H_1^*],$$

From relations (11) and (1) we get:

$$H_1(\bar{r}) = \frac{k^2 \cdot c_1}{2\pi jk} \cdot rot\left(\frac{e^{jkr}}{r} \cdot \bar{A}_0\right) = \frac{k^2 c_1}{2\pi jk}\,\bar{A}_0\, grad\left(\frac{e^{jkr}}{r}\right)$$

$$= \frac{k^2 \cdot c_1}{2\pi jk} \cdot \left(\frac{e^{jkr}}{r} jk - \frac{e^{jkr}}{r^2}\right) \cdot \bar{\delta}_0 \cdot \bar{A}_0 \cong \frac{k^2 \cdot c_1 \cdot e^{jkr}}{2\pi r} \cdot \bar{\delta}_0 \cdot \bar{A}_0 \quad (12)$$

neglecting the term $\frac{e^{jkr}}{r^2}$ in accordance with the assumption made above. Substituting (11) and (12) in the relation which gives the flux density $\bar{S}$ we get:

$$\bar{S} = \frac{c \cdot k^4}{32\pi^2 \cdot r^2} c_1 \cdot c_1^* A_0 \cdot (\bar{\delta}_0 \bar{A}_0) =$$

$$= \frac{c \cdot k^4}{32\pi^2 r^2} \cdot c_1 \cdot c_1^* \left[\bar{\delta}_0 \cdot (\bar{A}_0 \cdot \bar{A}_0) - \bar{A}_0 \cdot (\bar{\delta}_0 \cdot \bar{A}_0) \right]$$

Towards the direction $\bar{\delta}_0$ the flux density will be equal to:

$$S_\delta = \bar{S} \cdot \bar{\delta}_0 = \frac{c \cdot k^4 c_1 c_1^*}{32\pi^2 r^2} \left[A_0^2 - (\bar{\delta}_0 \cdot \bar{A})^2 \right] = \frac{c \cdot k^4 \cdot \sin^2 x}{32\pi^2 \cdot r^2} \cdot c_1 \cdot c_1^*$$

where x is the angle between $\bar{A}_0$ and $\bar{\delta}_0$ and $\bar{A}_0{}^2 = 1$

Substituting the value of c_1 from (11)', we get:

$$S_\delta = \frac{c \cdot k^4 \cdot \sin^2}{32\pi^2 \cdot r^2} \int_V \int_V n_1(\bar{r}_1) \cdot n_2(\bar{r}_2) \cdot e^{j(\bar{k} - k\bar{\delta}_0)(\bar{r}_1 - \bar{r}_2)} \cdot dV_1 dV_2 \quad (12')$$

The quality S_δ has "random" distribution.

The mean value of it will be:

$$\boxed{\bar{S}_\delta = \frac{c \cdot k^4 \cdot \sin^2}{32 \cdot \pi^2 \cdot r^2} \int_V \int_V \overline{n_1(\bar{r}_1) \cdot n_2(\bar{r}_2) \cdot e^{j(\bar{k} - k\bar{\delta}_0) \cdot (\bar{r}_1 - \bar{r}_2)}} \cdot dV_1 dV_2 \quad (13)}$$

From (13) we observe that the mean value of the energy flux density at the point of observation is expressed as a function of the correlation coefficient of Volume B $(\bar{r}_1, \bar{r}_2)$ of the refraction index fluctuation.

We assume at first that the turbulence inside the volume V is homogeneous, i.e. that:

$$B(\bar{r}_1, \bar{r}_2) = B(\bar{r}_1 - \bar{r}_2)$$

(The homogeneous turbulence is also isotropic as long as $B(\bar{r}_1 - \bar{r}_2) = B(|\bar{r}_1 - \bar{r}_2|)$.

If $\bar{\rho} = \bar{r}_1 - \bar{r}_2$ and using the fact that the function $B(\bar{\rho})$ and the function of the phase distribution of the atmospheric nonhomogeneity $\Phi(\bar{u})(u_i = \frac{2\pi}{\ell_i})$ constitute the first function $B(\rho)$ the Fourier transformation of $\Phi(u)(u_i = \frac{2\pi}{\ell_i})$ and the opposite*, i.e.

* See e.g. I. Bendat "Principles and Applications of Random Noise Theory", ch. II. See also the attached Appendix, ℓ_1 represents the "Scale of Turbulence", of order i (see Fig. 2).

$$f(\tau) = \left[\frac{\sin(\tau\alpha)}{(\pi\tau)}\right]^3$$

Zero crossing: $\tau_o = \frac{K\pi}{\alpha} \; (0 \pm 1 \pm 2 \ldots)$

Maxima: $\tau_m = \chi_v/\alpha$

(where χ_v solution of $\chi_v = \tan \chi_v$)

Maxima values:

$$F(\tau_m) = \left[\frac{\alpha}{\pi} \frac{1}{\sqrt{1+\chi_v^2}}\right]^3$$

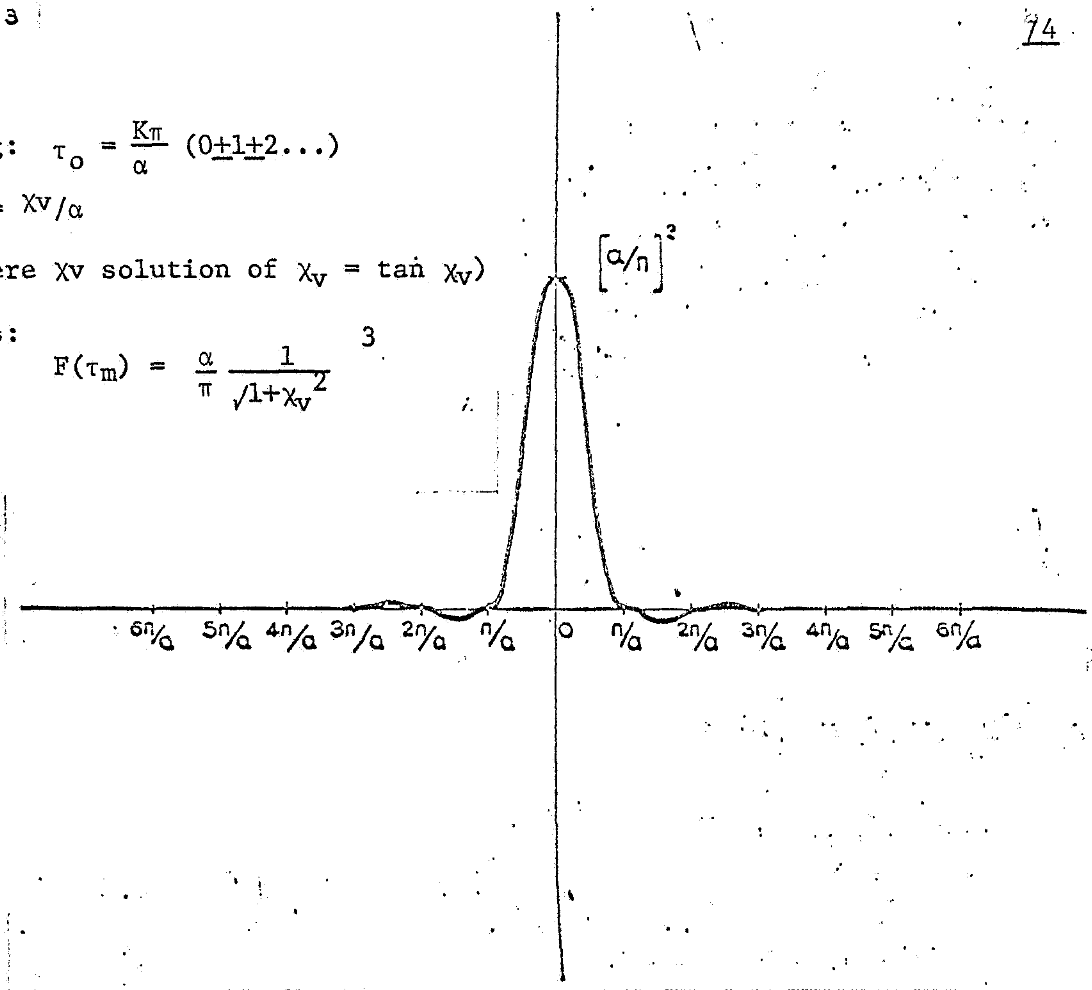

Fig. 1

$B(\bar{\varrho}) = \iiint_{-\infty}^{\infty} e^{-j\bar{u}\cdot\bar{\varrho}} \Phi(\bar{u})\, d\bar{u},$ referring to the integral (13)

$I = \frac{1}{8\pi^3} \int_v B(\bar{\varrho}) \cdot e^{j[(\bar{k} - k\bar{\delta}_0)\cdot\bar{\varrho}]} dV_\varrho$ we get:

$$I = \iiint_{-\infty}^{\infty} \Phi(\bar{u}) \cdot d\bar{u} \cdot \frac{1}{8\pi^3} \int_v e^{j[(\bar{k} - k\bar{\delta}_0 - \bar{u})\cdot\bar{\varrho}]} dV_\varrho$$

Let us examine briefly the integral:

$$F(\bar{\tau}) = \frac{1}{8\pi^3} \cdot \int_V e^{j\bar{\tau}\bar{\varrho}} \, dV_\varrho$$

Since the volume of integration has infinite dimensions $F(\bar{\tau}) = \delta(\bar{\tau})$*, then $I = \Phi(\bar{k} - k\bar{\delta}_o)$.

If the volume V has infinite dimensions the function $F(\bar{\tau})$ gives:

$$\iiint_{-\infty}^{\infty} F(\bar{\tau}) \, . \, d\bar{\tau} = \int_V \delta(\bar{\varrho}) \, . \, dV_\varrho = 1 \, ;$$

Figure 1 shows abrupt maximum near the value of $\tau = 0$ and outside the zero region oscillates decreasing rapidly**.

Further, since $F(o) = \frac{V}{8\pi^3}$, and the function $F(\tau)$ is basically limited inside the volume of order $\frac{8\pi^3}{V}$ we will have:

$$I = \iiint_{-\infty}^{\infty} \Phi(\bar{u}) \, . \, F(u - \bar{k} + k\bar{\delta}_o) \, . \, d\bar{u} \cong \iiint_{V/8\pi^3} \Phi(\bar{u}) . \frac{V}{8\pi^3} \, d\bar{u}$$

$$= \frac{V}{8\pi^3} \iiint_{V/8\pi^3} \Phi(\bar{u}) \, . \, d\bar{u}$$

where the integration takes place inside the volume $\frac{8\pi^3}{V}$ which contains the point $\bar{u} = \bar{k} - k\bar{\delta}_o$. Therefore, $I = \Phi_o\,(\bar{k} - k\bar{\delta}_o)$ where Φ_o represents the mean value of $\Phi(\bar{u})$ (which has no relation whatsoever to the statistical mean value), inside the above volume.

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* $\delta(\bar{\tau})$ the delta function of Dirac. This must not be confused with the unit vector δ_o.

** Let us consider, e.g. that the volume V is a cube of side 2a. We will have:

$$F(\tau) = F(\tau_1)\, F(\tau_2)\, F(\tau_3) = \frac{\eta\mu\tau_1 . a}{\pi . \tau_1} \cdot \frac{\eta\mu\tau_2 . a}{\pi . \tau_2} \cdot \frac{\eta\mu\tau_3 . a}{\pi . \tau_3} \, ;$$

where $\bar{\tau} = (\tau_1, \tau_2, \tau_3)$.

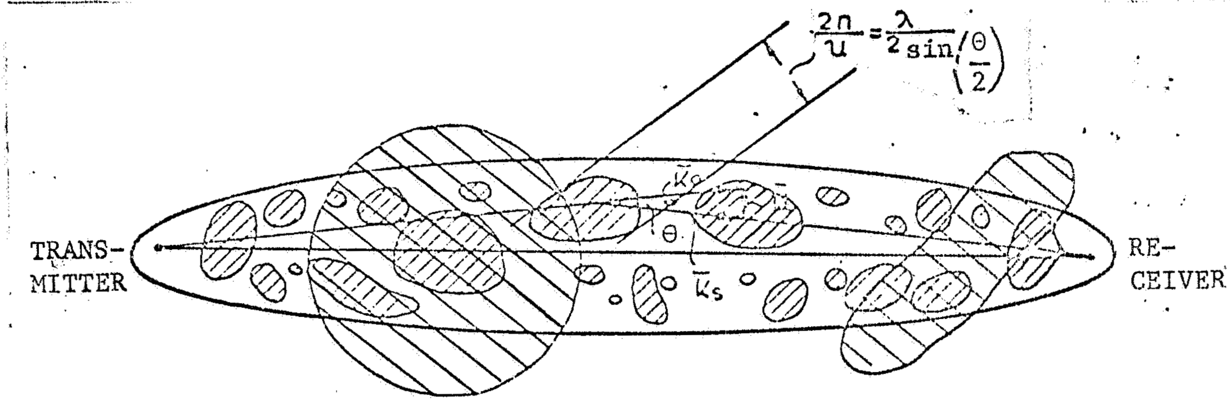
Each of the terms of the above product represents the spectrum $\frac{1}{2\pi}\int_{-a}^{a} e^{j\tau . x} \, . \, dx$ of the plain wave $e^{j\tau . x}$, of sinusoidal train length 2a and wave length $\lambda_o = \frac{2\pi}{\tau_o}$ (in Fig. 1, 0 is being taken in the value τ_o).

(continued)

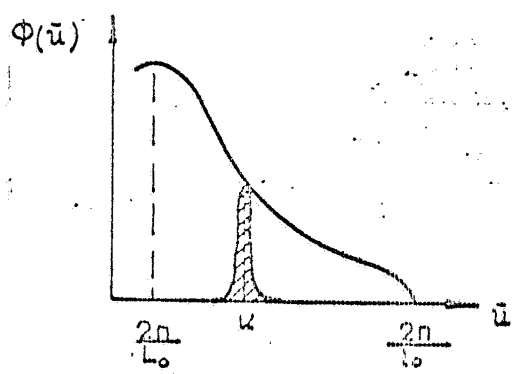
From (13) already it follows that:

$$\overline{S}_{\delta} = \frac{c.k^{4}.V.\frac{\sin}{x}}{4r^{2}}.\Phi(\overline{k}-k.\overline{\delta}_{0}) \qquad (14)$$

/5



The vector $\bar{u} = \bar{k}_o - \bar{k}_s$ ($|\bar{K}_o| \cong |\bar{K}_s|$) is directed practically vertical. Due to this, if the spectral distribution $\Phi(u)$ of the atmospheric nonhomogeneities is being explored by experimental presentation of the propagation mechanism for each case, this is always referred to the vertical distribution of the above nonhomogeneities. Therefore, from the receiver's point of view, the distinction between statistically isotropic and non-isotropic media is very difficult.



L_o outer scale of turbulence

ℓ_o inner scale of turbulence

Fig. 2

**Cont'd from p. 10

For $\tau = o$, $F(o) = (\frac{\alpha}{\pi})^2$, for $\tau_o = \frac{\pi}{\alpha}$, $F(\tau) = 0$ and for large values τ the $F(\tau)$ oscillates and rapidly approaches zero (Fig. 1). $\underset{(a \gg \lambda_0)}{a \to \infty}$ $F(\tau) = \delta(\tau)$.

Further, for the region contained between the first two zero crossings we have:

$$I = \int_{-\frac{\pi}{a}}^{\pi/a} \left(\frac{\sin(\tau)}{\pi\tau}\right)^2 d\tau = \frac{a^2}{4\pi^2}\left\{9\,S_i\,(3\pi) - 3\,S_i\,(\pi)\right\}$$

where $S_i(u) = \int_0^u \frac{\sin t}{t} dt$, $S_i(\pi) \doteq 1{,}85$, $S_i(3\pi) \cong 1{,}66$ and $I = 2{,}38\frac{a^2}{\pi^2}$

(continued)

If $\Phi(u)$ does not change substantially inside the volume $\frac{8\pi^3}{V}$, /6

$\Phi_o(\bar{k} - k\bar{\delta}_o) \sim \Phi(\bar{k} - k\bar{\delta}_o)$ then:

$$\bar{S}_\delta = \frac{c.k^4.V.\sin^2\chi}{4.r^2}\,\Phi(\bar{k} - k.\bar{\delta}_o) \qquad (15)$$

Relation (15) leads already to the solution of the equivalent transversal scattering cross section of the volume V: if this cross section is called σ, the element of scattering surface in the direction $\bar{\delta}_o$ inside the solid angle $d\Omega$ will be:

$$d\sigma = \frac{\bar{S}_\delta . r^2 . d\Omega}{\left(\frac{c.A_0^2}{8\pi}\right)} = \frac{\bar{S}_\delta . r^2 . d\Omega}{\left(\frac{c}{8\pi}\right)} =$$
$$= 2\pi k^4 . V . \sin^2\chi . \Phi(\bar{k} - k.\bar{\delta}_o)\, d\Omega \qquad (16)$$

From (16) it can be inferred that the scattering against an angle $\Phi(\sin\Phi = \frac{\bar{k}\delta_o}{k})$, is determined only from a very narrow region of the spectral distribution of the total scattered radiation, near the point $\bar{u} = \bar{k} - \bar{k}\bar{\delta}_o$. (See Fig. 2).

This means that only a small number (inversely proportional to the dimensions of the scattering volume) of monochromatic components composes the spectrum which results from scattering against the angle Φ. These components create diffraction fringes of constant distance determined from the relation:

$$l = \frac{2\pi}{|\bar{k} - k.\bar{\delta}_o|} = \frac{\lambda}{2\sin\frac{\Theta}{2}} \qquad (16)'$$

(verification of Bragg condition).

For the dimensions of the common volume a_i we will have:

$$l' = \frac{\lambda}{2\sin\frac{\Theta}{2} \pm \frac{\lambda}{a_i}}$$

(For: $\lambda << a_i$ and $\frac{\lambda}{a_i} << 2\sin\frac{\Theta}{2}$, $l \cong l'$)

**Cont'd fm pg. 10 & 11

while $$I_{o\lambda} = \int_{-\infty}^{\infty}\left(\frac{\sin\pi a)}{\pi\tau}\right)^2 d\tau = \frac{a^2}{4\pi^2}\cdot\left\{9\int_0^{\infty}\frac{\sin x}{x}dx - 3\int_0^{\infty}\frac{\sin x}{x}dx\right\} = \frac{a^2}{4\pi^2}.6\frac{\pi}{2} = 2{,}36\frac{a^2}{\pi^2}$$

It follows that volume $\frac{8\pi^3}{8\alpha^3} = \frac{8\pi^3}{V}$ includes $\sim$ 99% of $F(\tau)$, as asserted.

Experimentally so, if the geometry of coupling between transmitter and receiver is known (Φ is known) and if the phenomenon is considered to be statistical, placing two antennas in position of reception apart by

$(2v + 1)\frac{\ell'}{2}$ (so that when one receives maximum the other receives minimum signal) we can determine the dimensions of the common scattering volume in the horizontal and the vertical direction

Because: $\bar{k} - k\bar{\delta}_0 = 2k \cdot \sin\frac{\Theta}{2} = \frac{4\pi}{\lambda}\sin\frac{\Theta}{2}$ we have finally: (given that $x = \frac{\pi}{2} \pm \Phi$) .

$$d\sigma = 2\pi\left(\frac{2\pi}{\lambda}\right)^4 \cdot V \sin\Theta \cdot \Phi\left(4\pi/\lambda \sin\frac{\Theta}{2}\right) d\Omega \qquad (17)$$

Given that, as a rule, the angle Φ is very small: $\cos^2\Phi \cong 1$ therefore:

$$\boxed{d\sigma = 2\pi\cdot\left(\frac{2\pi}{\lambda}\right)^4 \cdot \Phi\left(\frac{4\pi}{\lambda}\sin\frac{\Theta}{2}\right)} \qquad (18)$$

represents the equivalent scattering surface for a unit of solid angle, unit volume and unit density of the oncoming power on the volume of diffusion.

We remind you that

$$\Phi(\bar{k}) = \frac{1}{(2\pi)^3} \cdot \int_V B(\bar{r}) \cdot e^{j\bar{k}\bar{r}}\, d\bar{r} \qquad (18)'$$

and that $\bar{k} = \bar{k}_0 - \bar{k}_1$ where $\bar{k}_0$ is the direction of the oncoming radiation S on the volume of scattering and $\bar{k}_1$, the direction of the scattered beam against an angle Φ relative to the penetrating direction.

Meaning:

$$\Phi(\bar{k}_0 - \bar{k}_1) = \Phi\left(\frac{4\pi}{\lambda}\sin\frac{\Theta}{2}\right) = \frac{1}{(2\pi)^3}\int_V B(\bar{r}) \cdot e^{j(\bar{k}_0 - \bar{k}_1)\bar{r}}\, d\bar{r}$$

Substituting into (18) we get:

$$d\sigma = \frac{4\pi^2}{\lambda^4}\left|\int_V B(\bar{r}) \cdot e^{j(\bar{k}_0 - \bar{k}_1)\bar{r}}\, d\bar{r}\right| \qquad (19)$$

Let us consider now the relation (12)'. This is written:

$$S_\delta = \frac{c \cdot k^4 \sin^2 x}{32\pi^4 r^2} \int_V n_1(\bar{r}_1) \cdot e^{j(\bar{k} - k\bar{\delta}_0)\bar{r}_1} \cdot d\bar{r}_1 \int_V n_1(\bar{r}_2) \cdot e^{j(\bar{k} - k\bar{\delta}_0)\bar{r}_2} \cdot d\bar{r}_2 =$$

$$= \frac{c\, k^4 \sin^2 x}{32\pi^8 r^2} \left| \int_V n(\bar{r}) \cdot e^{j(\bar{k}_0 - \bar{k}_1)\bar{r}} \cdot d\bar{r} \right|^2$$

and

$$d\sigma_0 = \frac{S_\delta \cdot r^2 \cdot d\Omega}{c/8\pi} = \frac{k^4 \sin^2 x \, d\Omega}{4\pi^2} \left| \int_V n(\bar{r}) \cdot e^{j(\bar{k}_0 - \bar{k}_1)\bar{r}} \cdot d\bar{r} \right|^2$$

and because $\sin^2 x \cong 1$ the per unit of solid angle equivalent scattering cross section will be:

$$d\sigma_0 = \frac{4\pi^2}{\lambda^4} \cdot \left| \int_V n(\bar{r}) \cdot e^{j(\bar{k}_0 - \bar{k}_1)\bar{r}} \, d\bar{r} \right|^2 \qquad (20)$$

Relation (20) constitutes equivalent expression of (19).

It remains now to examine the nature of the potential, which the distribution $n(\bar{r})$ is due to. We will then examine the reason and the mechanism by which, inside atmospheric volume V, turbulence is created and developed, a result of which is the distribution $n(\bar{r})$*.

Let us consider originally that inside the above volume the viscous fluid (atmospheric air) has laminar flow. This flow is characterized by the value of the parameters $\frac{\mu}{\rho}$ (where μ is the viscosity of the fluid and ρ its density), which gives the measure of the retarding forces under which the molecules of the fluid are, given: its viscosity, the flow velocity V and the length L_o characterizing the dimensions between the limiting surfaces of the volume V, inside which the flow is laminar (e.g. if the flow takes place inside a pipe, L_o represents the diameter of the pipe).

The laminar flow of the fluid is "stable" only in the case where Reynolds number $R = \frac{v \cdot L_0}{\left(\frac{\mu}{\rho}\right)}$ does not exceed some critical value R_c. /7

While R increases (due to an increase in the flow velocity, or an increase in the diameter of the pipe of flow, or a decrease in the viscous forces of the fluid) the laminar flow becomes unstable due to the following reasons:

Let us consider that for some reason a "clot" is created inside the fluid which is characterized by velocity variation v' inside the region of diameter L_o. The "period" $\tau = \frac{\ell}{v'}$ characterizes the time required for the above change to occur. The per unit of mass energy of fluctuation will be v'^2. It

*See Appendix.

follows that the per unit of time energy, which is transferred by the laminar form of flow in the turbulent movement will be equal to $\frac{v'^2}{\tau} \doteq \frac{v'^3}{\ell}$. On the other hand, the per unit of mass of fluid and the per unit of time dissipated in heat energy (due to friction inside the turbulent clot) will be equal to:

$$\left(\frac{\mu}{\varrho}\right) \cdot (grad v')^2. \qquad \text{But} \qquad grad v' = \frac{v'}{l}.$$

As a result of the above, the condition for the resulting velocity change to remain stable is:

$$\frac{v'^3}{l} > \left(\frac{\mu}{\varrho}\right) \frac{v'^2}{l^2} \qquad \text{i.e.} \qquad R_l = \frac{l \cdot v'}{\left(\frac{\mu}{\varrho}\right)} > 1.$$

The above relation indicates that large scale disturbances occur easily. R_ℓ represents now "the inner Reynolds number" which is referred to more in the flow which is characterized by velocity v' and dimension ℓ.

While R_ℓ remains smaller than the value R_c for which $\frac{v \cdot L_0}{\left(\frac{\mu}{\varrho}\right)} \geqslant R_c$ the resulting fluctuations v' of scale ℓ change, of course, the nature of the original flow, but themselves remain "stable". Since now we increase the number $\frac{v \cdot L_o}{(\mu/\rho)}$ the fluctuations v' increase and the inner Reynolds number R_ℓ can exceed the critical value. This means that the "first order" fluctuations v' become unstable and transfer energy in other "second order" fluctuations v' of scale ℓ_1. As long as the number $\frac{v \cdot L_o}{(\mu/\rho)}$ increases further, the fluctuations now v'' "second class" become unstable transferring energy in third class fluctuations v'' of scale ℓ_2, etc.

Let ℓ_o be the range of the smaller (of higher order) fluctuations which can occur in the system under consideration and v_o their velocity.

Since for all lower order fluctuations the transfer energy from one range to the next was given partly as dissipated heat and partly as power $\frac{v_k^3}{\ell_k}$ for the "initiation" of the next order fluctuation, in the case (ℓ_o, v_o), the corresponding energy is being transformed purely into heat. In this case the per unit time energy which is transformed into heat will be

$$\varepsilon = \left(\frac{\mu}{\varrho}\right) \cdot \frac{v_0^2}{l_0^2}.$$

Given though that for all order disturbances of range $\ell > \ell_o$ the inner Reynolds number R_ℓ is large (for the troposphere $\backsim 10^6$), the percentage of energy, which is used each time from order to order, was minimal compared to the energy attributed to the "initiations" of the next class fluctuations. We can approximately say that the quantity $\frac{v'^3}{\ell}$ was transferred up to the last order almost untouched, and, therefore,

$$\frac{v'^3}{l} \cong \left(\frac{\mu}{\varrho}\right)\frac{v_0^2}{l_0^2} \quad \text{or} \quad v' \cong \sqrt[3]{\varepsilon \cdot l}$$

meaning that for all order fluctuations the velocities v' are proportional to the 1/3 power of the corresponding "scales of turbulence". From the relations

$$v_0 = \sqrt[3]{\varepsilon \cdot l_0} \quad \text{and} \quad \left(\frac{\mu}{\varrho}\right)\frac{v_0^2}{l_0^2} \cong \varepsilon$$

it follows also:

$$l_0 \cong \sqrt[4]{\frac{(\mu/\varrho)^3}{\varepsilon}} \quad (2) \qquad \text{and} \qquad v_0 \sim \sqrt[4]{\left(\frac{\mu}{\varrho}\right)\cdot\varepsilon} \quad (3)$$

The relations now between the maximum (L_o) and the minimum (ℓ_o) scale of turbulence as well as between the corresponding velocities of fluctuation v and v_o will be: [see (1), (2), (3)]

$$l_0 \cong \frac{L_0}{(R)^{3/4}} \quad \text{and} \quad v_0 \sim \frac{v}{(R)^{1/4}}$$

This means that as long as the Reynolds number of the flow as a whole inside the volume V increases (as long as, e.g. the wind velocity v increases), the minimum scale of turbulence becomes smaller. Given now that the electromagnetic radiation scattering is possible to happen for minimum wave length $\lambda \cong \ell_o$, it is obvious that tropospheric coupling of known wave length λ_i cannot physically be accomplished, even if high power transmission is used, beyond a distance for which the altitude of the common volume of scattering V, formed by the solid angles of transmission - reception, is situated in a region where $\lambda_i >> \ell_i$.

On the other hand it is obvious that the velocity v_i $(\bar{r})$ determines the distribution $n_i(\bar{r})$: if the coefficient of auto-correlation of the distribution n_i is on length r_i $\varrho(r_i) = \overline{n(r) \cdot n(r+r_i)}$ and the corresponding mean scale of turbulence

$$l_i = \int_0^\infty \varrho(r_i)\,dr_i \quad \text{is} \quad v_i \cong \sqrt[3]{\varepsilon \cdot l_i}$$

where v_i is the fluctuation velocity of the fluid that pertains to an atmospheric clot of diameter ℓ_i.

PART B.

DETERMINATION OF AMPLITUDE AND PHASE FLUCTUATIONS OF A PLANE MONOCHROMATIC WAVE THROUGH TURBULENT ATMOSPHERE

Introduction

The influence of turbulence in the lower atmospheric layers (troposphere) on microwave propagation causes scattering and fluctuations of the amplitude, phase, frequency and polarization of the microwaves. These phenomena are of great importance since on one hand they allow the prediction, through a statistical study of the quality of a radioelectric coupling through the troposhere, and on the other hand, they allow, by their proper interpretation, the introduction of essential conclusions regarding the nature of the propagation medium, the establishment of relationsships between radioelectric and meteorologic parameters, the knowledge of the degree of error of radioelectric directional aiming in the cases of satellite communications or in spacecraft general electronic guidance, the knowledge of the degree of error of telescopes or radiotelescopes, etc. /8

The problem is expressed as follows:

The radioelectric radius, from the transmitter to the receiver, follows a trajectory determined from the distribution n (x,y,z,t) (more accurately: from the first two factors of n relative to each variable) of the refraction index.

The transmitter and receiver may both be situated within the turbulence region or one within and the other without. In the first case we can substitute for the transmitter, an equivalent source located at the boundary surface of the turbulence volume (e.g. in the case of a satellite or a star being long distances from the Earth, it is possible to substitute for the transmitter, a plane wave on the separating surface marking the end of the troposphere). In the second case, we can generally disregard the portion of the turbulence volume behind the source since the influence of the scattered wave towards the rear of the transmitter (backscattering) is negligible. In our study we will consider the turbulence field to be "locally isotropic"*. Our problem is the determination of the statistical properties of the wave at a distance L from the source (transmitter). We will begin with a relatively simple example based on the geometric optics approximation.

I. EXAMPLE USING THE EQUATION OF GEOMETRIC OPTICS

1. As was developed in Part A, the electromagnetic wave scattering in a nonhomogenous atmosphere is described by Equation (4)

$$\nabla^2 E + k^2 . n^2 . \bar{E} + 2 \text{ grad } (\bar{E} \text{ grad } . \log n) = 0$$

*See Appendix.

We assume that the geometric dimensions of the nonhomogeneities in the volume distribution of the refraction coefficient are much larger than the wave length λ(i.e. $\lambda \ll \ell_o$ where ℓ_o is the "internal" scale of turbulence).

In this example, substituting in Eq. (4) $n = 1 + n_1$ and

$$\bar{E} = \bar{E}_0 + \bar{E}_1 , \quad (|E_1| \ll |E_0| , \ |n_1| \ll 1)$$

we have $\nabla^2 E_0 + k^2 E_0 = 0$ and

$$\nabla^2 E_1 + k^2 . E_1 + 2k^2 . n_1 . \bar{E}_0 + 2grad . (\bar{E}\ grad n_1) = 0$$

The last term of the above equation is approximately of the order of magnitude $kE \dfrac{n_1}{\ell_o}$ and the next to last term:

$$2 k^2 n_1 E_0. \quad \text{for } \lambda \ll l_0 .$$

or $kl_0 \gg 1$ is: $\quad 2k^2 n_1 |E_0| \gg k |E_0| \dfrac{n_1}{l_0} -$

$$- k |E| n_1/l_0$$

In addition, the term 2 grad [$\bar{E}$ grad log n] in Eq. (4) describes the variation of the plane of polarization, during the wave propagation, which for $\lambda \ll \ell_o$ is negligible. Under these conditions Eq. (4) becomes:

$$\nabla^2 \bar{E} + k^2 . n^2.(\bar{r}) . \bar{E} = 0 \tag{4.1}$$

For each component of $\bar{E}$ in the direction x,y,z, we have the relation:

$$\nabla^2 u + k^2 . n^2 (\bar{r}) . u = 0 \tag{4.2}$$

or

$$\nabla^2 u/u + k^2 . n^2 (\bar{r}) \equiv \nabla^2 \log u + (\nabla \log u)^2 + + k^2 . n^2 (\bar{r}) = 0 \tag{4.3}$$

We apply $u = A . e^{i\phi}$ where: A is the amplitude and ϕ, the phase of the component under consideration.

Substituting in Eq. (4.3) and equating to zero the real and imaginary parts we have:

$$\nabla^2 \log A + (\nabla \log A)^2 - (\nabla\varphi)^2 + k^2 n^2 (\bar{r}) = 0 \tag{4.4}$$

and

$$\nabla^2 \varphi + 2 \nabla \log A . \nabla\varphi = 0 \tag{4.5}$$

First we observe that since $\phi = \bar{k}\ \bar{r}$ and $|\nabla\phi| = k$, the last two terms of Eq. (4.4) are of the order of $k^2 = \frac{4\pi^2}{\lambda^2}$. Further, we consider that the amplitude A varies substantially in instances of the order ℓ_o. Hence the expression:

$\nabla^2 \log A + (\nabla \log A)^2 = \frac{\nabla^2 A}{A}$ is approximately of the order of $\frac{1}{\ell_o^2}$, since $\lambda \ll \ell_o$, $\frac{4\pi^2}{\lambda^2} \gg \frac{1}{\ell_o^2}$ and, therefore, the first two terms of Eq. (4.4) can be disregarded in comparison with the last two terms. Eq. (4.4) then becomes:

$$(\nabla\phi)^2 = k^2 \cdot n^2 (\bar{r}) = 0 \qquad (4.6)$$

To determine now logA and ϕ we will use the system of Eqs. (4.5) and (4.6), i.e. the system of geometric optics equations (ch. 2).

We assume that the distribution $(\bar{r})$ is random in space

$n(\bar{r}) = 1 + n_1(\bar{r})$ and $|n_1(\bar{r})| \ll 1$

(see Part A.)

We put $\phi = \phi_o + \phi_1$ and $\log A = \log A_o + x$

where $x = \log \frac{A}{A_o}$ /9

represents the amplitude fluctuations on a logarithmic scale.

The Eqs. (4.5) and (4.6), therefore, take the form:

$$(\nabla\varphi_0)^2 + 2 \cdot \nabla\varphi_0 \cdot \nabla\varphi_1 + (\nabla\varphi_1)^2 = k^2 + 2k^2 \cdot n_1(\bar{r}) + k^2 \cdot n_1^2(\bar{r}) \qquad (4.7)$$

$$\nabla^2\varphi_0 + \nabla^2\varphi_1 + 2 \cdot \nabla \log A_0 \cdot \nabla\varphi_0 + 2\nabla \log A_0 \nabla\varphi_1 + 2\nabla x \nabla\varphi_0 + 2\nabla x \nabla\varphi = 0 \qquad (4.8)$$

but: $(\nabla\varphi_0)^2 - k^2$ and the expression: $\nabla^2\varphi_0 + 2\nabla \log A \cdot \nabla\varphi_0$ is of "zero" order

As a result of Eqs. (4.7) and (4.8) become:

$$\nabla\varphi_1 \cdot (2\nabla\varphi_0 + \nabla\varphi_1) = 2k^2 \cdot n_1(\bar{r}) + k^2 \cdot n_1^2(\bar{r}) \qquad (4.9)$$

and $$\nabla^2\varphi_1 + 2\nabla \log A_0 \cdot \nabla\varphi_1 + 2\nabla x \cdot \nabla\varphi_0 + 2\nabla x \nabla\varphi_1 = 0 \qquad (4.10)$$

Now, in the case where $|\nabla\phi_1| \ll |\nabla\phi_o| = k$, i.e. $\lambda |\nabla\phi_1| \ll 2\pi$ we can omit the term $(\nabla\phi_1)^2$ in Eq. (4.9). Furthermore, the term $k^2 n_1^2(\bar{r})$, $(|n_1(\bar{r})| \ll 1)$. can be omitted from the right-hand side of Eq. (4.9).

Then Eq. (4.9) becomes:

$$\nabla\phi_o \cdot \nabla\phi_1 = k^2 n_1 (\bar{r}) \tag{4.11}$$

valid for $\lambda |\nabla\phi_1| << 2\pi$, i.e. under the condition that the phase varies negligibly in length equal to λ.

As result, we can also omit the last term of Eq. (4.10), and then:

$$\nabla^2\phi_1 + 2\nabla \log A_o \nabla\phi_1 + 2\nabla x \cdot \nabla\phi_o = 0 \tag{4.12}$$

Let us consider now a plane wave in the direction of axis x. We have $\phi_o = k.x$ and $A = \text{const}$. Eqs. (4.11) and (4.12) take the form

$$\frac{\partial\phi_1}{\partial x} = k \cdot n_1 (\bar{r}) \tag{4.13}$$

and

$$\nabla^2\phi_1 + 2k \frac{\partial x}{\partial x} = 0 \tag{4.14}$$

Let us consider that with the transmitter at the point 0, y, z and the receiver at the point L, y, a, we have, by integration of Eqs. (4.13) and (4.14),

$$\varphi_1 (L,y,z) = k\int_0^L n_1 (x, y, z)\, dx \tag{4.15}$$

and $$x (L, y, z) = -\frac{1}{2k}\left[\left(\frac{\partial\varphi_1}{\partial x}\right)_{(L,y,z)} - \left(\frac{\partial\varphi_1}{\partial x}\right)_{(o,y,z)} + \int_0^L \left(\frac{\partial^2\varphi_1}{\partial y^2} + \frac{\partial^2\varphi_1}{\partial z^2}\right) dx\right]$$

But the quantity

$$\frac{1}{2k}\left[\left(\frac{\partial\varphi_1}{\partial x}\right)_{(L,y,z)} - \left(\frac{\partial\varphi_1}{\partial x}\right)_{(o,y,z)}\right] =$$

$$= \frac{1}{2}\left[n_1 (L, y, z) - n_1 (o, y, z)\right]$$

is smaller than the integral

$$\frac{1}{2k}\int_0^L \left(\frac{\partial^2\varphi_1}{\partial y^2} + \frac{\partial^2\varphi_1}{\partial z^2}\right) dx$$

Therefore, we can approximately apply:

$$x(L, y, z) \cong -\int_{0}^{L} dx \int_{0}^{x} \left[\frac{\partial^2 n_1(\xi, y, z)}{\partial y^2} + \frac{\partial^2 n_1(\xi, y, z)}{dz^2} \right] d\xi$$

and, in the case of isotropic turbulence,

$$x(L, y, z) = -\int_{0}^{x} dx \int_{0}^{L} \frac{\partial^2 n_1(\xi, z)}{\partial z^2} d\xi \qquad (4.16)$$

From relations (4.15) and (4.16), and knowing the function $n_1(x,y,z)$ or in the case of isotropic atmosphere $n_1(x)$, the functions ϕ_1 (L) and $\dot{x}(L)$ are calculated.

Before proceeding with the general solution of the wave equation, we should refer briefly to the criteria which determine the distance L, so the approximation of the geometric optics by which the solutions (4.15) and (4.16) were accomplished can be taken into account without appreciable error.

2. Determination of the limit of application of the geometric optics equations as applied to the electromagnetic radiation propagation through turbulent atmosphere.

2.1 The theory of the amplitude and phase variation of electromagnetic waves traveling through turbulent atmosphere developed in Chapter 1 was based on the equations of optical geometry which, as we proved, resulted from the wave equation for: $\lambda \ll \ell_0$.

The above condition is not sufficient since we consider that each atmospheric "clot" of the nonhomogeneous troposphere produces significant diffraction of the incident wave.

Let us examine the principal factors of the subject:

From the definition of a monochromatic plane wave, this type of wave has constant amplitude universally in space and time.

Any wave, therefore, the amplitude of which does not precisely fulfill the above conditions, is more or less nonmonochromatic.

Let us then examine the problem of the "degree of monochromatic character" of a wave, i.e. the determination of the limit of application of the geometric optics equations.

a) Let us consider an electromagnetic wave, the amplitude of which is a function of time universally in space.

Let ω_0 be the angular "mean" frequency of the wave.

Suppose the component of the electric field of the wave has the form of $E(t).e^{j\omega_0 t}$. This field (nonmonochromatic) can be analyzed in monochromatic components using its Fourier integral.

The amplitude of a frequency component ω will accordingly be equal to

$$\int_{-\infty}^{\infty} E_0(t) \cdot e^{j(\omega-\omega_0)t} \, dt \qquad \text{(I)}$$ /10

The factor $e^{j(\omega-\omega_0)t}$ is a periodic function of average value equal to zero. If the E_0 is constant, the above integral will be zero for all values of $\omega \neq \omega_0$. If now $E_0(t)$ is variable but varies "little" in a time interval of the order of $\frac{1}{\omega-\omega_0}$, the integral differs from zero significantly less, as the variation of E_0 in the above time interval is smaller.

In order for the value (I) to be distinctly different from zero, it is necessary that E_0 vary substantially in the time interval $\frac{1}{\omega-\omega_0}$.

Let Δt be the above time interval.

If we designate $\Delta\omega$ the frequency interval (about the value ω_0) which enters into the spectral determination of the wave, we have:

$$\Delta\omega \; \Delta t \cong 1 \qquad \text{(II)}$$

We observe, i.e. that a wave is significantly more monochromatic ($\Delta\omega$) as the time interval Δt in which the amplitude $E_0(x_0, y_0, z_0)$ registers a given change is longer.

b) Let Δx, Δy, Δz be the distances in the direction of the axes x,y,z where the amplitude $E_0(t_0)$ changes (substantially). In a given time t_0 the field as a function of x (for x,y ; const.) has the form

$$E_0(x) \cdot e^{jk(x).x} \quad \text{where} \quad k(x) = \frac{2\pi}{\lambda(x)} .$$

By exactly similar deduction as in 2.1 a, we develop the relations:

$$\Delta k(x) \cdot \Delta x \cong 1, \quad \Delta k(y) \cdot \Delta y \cong 1$$
$$\text{and} \quad \Delta k(z) \cdot \Delta z \cong 1 \qquad \text{(III)}$$

From the above relations, it is obvious that if we consider an electromagnetic nonmonochromatic beam to be of finite cross section, its direction of propagation cannot be constant.

Considering the axis x as (mean) axis of direction of the beam, we define the angle Θ_y of the divergence of the beam by the relation:

$$\Theta_y \cong \frac{1}{k.\Delta y} \cong \frac{\lambda}{\Delta y} \tag{IV}$$

Therefore, a nonmonochromatic convergent beam does not produce one bright point on a screen, as in geometric optics, but a spot of diameter

$\Delta \approx \frac{1}{k.\Theta} \approx \frac{\lambda}{\Theta}$. This signifies that for the observer, it is not possible to distinguish an obstacle between the source and the screen of diameter $\ell_o < \Delta \sim \frac{\lambda}{\Theta}$.

Inversely:

For a monochromatic radiation propagating through obstacles of minimum diameter, ℓ_o (in this case ℓ_o is the minimum or "internal" scale of turbulence) at the distance L from one of the above obstacles (atmospheric clots), the diameter of the scattered beam will be equal to $\Theta . L \quad \frac{\lambda.L}{\ell_o}$. In order for the geometric shadow of the obstacle ℓ_o to remain relatively uniform (condition of geometric optics), the following relation must be fulfilled:

$\lambda L/\ell_o \ll \ell_o$ or for given λ, ℓ_o the distance L between transmitter and receiver for which, in a turbulent atmosphere, the propagation occurs, is, according to the laws of geometric optics, limited by the relation:

$$L \ll L_{cr} = \frac{\ell_o^2}{\lambda} \tag{V}$$

e.g. for $\ell_o \approx 5$ m and $\lambda = 10$ cm

we have $L_{cr} \cong 250$ m

II. 1. SOLUTION OF THE WAVE EQUATION BY THE METHOD OF "SMALL DISTURBANCES".*

At long distances, $\left(L > \frac{\ell_o^2}{\lambda}\right)$, from the transmitter, we cannot disregard the diffraction of the wave by the atmospheric nonhomogeneities ℓ_o.

For a general solution, therefore, of the problem we must start with the general equation:

$$\nabla^2 u + k^2 . n^2 (r) u = 0 \tag{5}$$

We apply as in Ch. I

*Born's approximation. See reference and Mandl, "Quantum Mechanics".

$$n(\bar{r}) = 1 + n_1(\bar{r}) \tag{5.1}$$

where $|n_1(\bar{r})| << 1.$

The principle of the method of "small disturbances" consists in considering the component u of the electric field to be calculated, as a sum of one term u_o representing the field in the absence of turbulence and of another term u_1 representing the influence of turbulence on the term u.

$$u = u_o + u_1 \tag{5.2}$$

Substituting Eqs. (5.1) and (5.2) in Eq. (5) and considering that:

$$\nabla^2 u_o + k^2 u_o = 0 \tag{5.3}$$

we have
$$\nabla^2 u_1 + k^2 u_1 + 2n_1 k^2 (u_o + u_1) + k_1^2 n_1^2 (u_o + u_1) = 0 \tag{5.4}$$

The last term of Eq. (5.4), of order n_1^2, can be omitted.

If we now suppose that $|u_1| << |u_o|$* or more exact $|\frac{u_1}{u_o}| \sim n_1$, we can omit the term $n_1(\bar{r}) \cdot u_1$ in Eq. (5.4).

Therefore, we have:

$$\nabla^2 u_1 + k^2 u_1 = -2k^2 \cdot n_1(\bar{r}) \cdot u_o \tag{5.5}$$

Let now: $u_o = A_o e^{j\phi o}$ and $u = A.e^{j\phi}$

is: $\log u = \log A + j\phi = \log(u_o + u_1) =$ /11

$$= \log u_o + \log\left(1 + \frac{u_1}{u_o}\right)$$

But according to the assumptions made of "weak" influence of turbulence,

$$\log\left(1+\frac{u_1}{u_o}\right) \sim \frac{u_1}{u_o} \quad \text{and}$$

$$\log A + j\varphi = \log A_o + j\varphi + \frac{u_1}{u_o}$$

whence:

$$\log\frac{A}{A_o} = x = Re\left[\frac{u_1}{u_o}\right]$$

and
$$\varphi - \varphi_o = \varphi_1 = Im\left[\frac{u_1}{u_o}\right] \tag{5.6}$$

*This is especially valid for couplings of line of sight contact.

2. Solution of the wave equation by the method of "smooth disturbances".

The equation (5.5) and formulas (5.6) are valid for

$$|x| << 1 \quad \text{and} \quad |\phi_1| << 1$$

However, the above conditions are more disciplined under the condition $\lambda\ |\nabla\phi_1| << 2\pi$, which we considered in the solution by the geometric optics approximation.

In order to avoid the distinction between boundary conditions of cases I and II, it is preferable to express the wave equation in the form:

$$\frac{\nabla^2 u}{u} + k^2\, n^2\, (\bar{r}) = \nabla^2 \log u + (\nabla \log u)^2 + k^2\, n^2\, (\bar{r}) = 0 \qquad (5.7)$$

and to apply to it the method of "small disturbances".

Applying:

$$\log u = A + j\varphi = \psi \quad \{ Re[\psi] = \log A,\ Im[\psi] = \varphi \}$$

which is $\nabla^2 \psi + (\nabla\psi)^2 + k^2 [1 + n_1(\bar{r})]^2 = 0$ (5.8)

Substituting: $\psi = \psi_0 + \psi_1$ where ψ_0 satisfies the condition: $\nabla^2\psi_0 + (\nabla\psi_0)^2 + k^2 = 0$ (5.9)

Substituting in Eq. (5.9) we get:

$$\nabla^2\psi_1 + \nabla\psi_1 (2\nabla\psi_0 + \nabla\psi_1) + 2k^2.n_1(r) + k^2.n_1^2(\bar{r}) = 0 \qquad (5.10)$$

omitting the term $k^2 . n_1^2$ (r) and, under the condition $|\nabla\psi_1| << |\nabla\psi_o|$, the term $(\nabla\psi_1)^2$ we get

$$\nabla^2\psi_1 + 2\nabla\psi_0\, \nabla\psi_1 + 2k^2.\, n_1(\bar{r}) = 0 \qquad (5.11)$$

Equation (5.11) is valid under the conditions

$$|n_1(\bar{r})| << 1 \text{ and } |\nabla\psi_1| << |\nabla\psi_0| \text{ because}$$
$$|\nabla\psi_0| \sim k = \frac{2\pi}{\lambda},\ \lambda\, |\nabla\psi_1| << 2\pi$$

a condition which expresses the stability of ψ_1 on a distance of the order λ.

Applying $\psi_1 = e^{-\psi_0} . w$ (5.12)

we transform Eq. (5.11) into the form

$$\nabla^2 w + k^2 w + 2k^2 . n_1 (r) . e^{\psi_0} = 0 \qquad (5.13)$$

Since $e^{\psi_o} = u_o$ we see that Eq. (5.13) coincides with Eq. (5.5) which results from the method of "small disturbances".

Because $\psi = \log A + j\varphi$ & $\psi_0 = \log A_0 + j\varphi_0$

are $\psi_1 = \psi - \psi_0 = \log \frac{A}{A_0} + j(\varphi - \varphi_0) = x + j\varphi$

and

$$\log A/A_0 = x = Re\,[\psi_1] = Re\left[\frac{u_1}{u_0}\right]$$
$$\varphi - \varphi_0 = \varphi_1 = Im\,[\psi_1] = Im\left[\frac{u_1}{u_0}\right] \tag{5.14}$$

Equation (5.5) and relations (5.6) agree with Eqs. (5.13) and (5.14) respectively.

The conditions for this are: $\lambda|\nabla\phi_1| << 2\pi$ and $\lambda|\nabla x| << 1$ and not directly $|\phi_1| << |\phi_o|$ and $|x| << 1$.

After deriving the identity of the boundary conditions for the solution of the wave equation by approximation I and the general case II, we arrive at the solution of Eq. (5.13). It is known* that this solution will be given by the relationship:

$$w(\bar{r}) = \frac{k^2}{2\pi} \int_V n_1(\bar{r}')\, u_0(\bar{r}')\, \frac{e^{jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\, dV' \tag{5.15}$$

where $\overline{r'}$ is a variable vector extended to each of the points of the volume V, and $\bar{r}$, a vector extended to the point of observation (receiver).

The integration is extended on the entire volume, inside which:

$n_1(\bar{r}) \neq 0$. The quantity $\psi_1 = \frac{w(\bar{r})}{u_o(\bar{r})}$ will be given by the relationship:

$$\psi_1(\bar{r}) = \frac{k^2}{2\sin_o(\bar{r})} \cdot \int_V n_1(\bar{r}')\, u_0(\bar{r}')\, \frac{e^{jk(\bar{r}-\bar{r}')}}{|\bar{r}-\bar{r}'|}\, dV \tag{5.16}$$

This function constitutes the general solution of (5.11) for all values of ψ_o which satisfy the equation:

$$\nabla^2\psi_0 + (\nabla\psi_0)^2 + k^2 = 0$$

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*For a very detailed analysis see Mandl, "Quantum Mechanics".

Let us consider now a plane monochromatic wave traveling along one of the axes, say x.

This will be $u_o(r) = A_o\, e^{jkx}$ and then (5.16) gives:

$$\psi_1(\bar{r}) = \frac{k^2}{2\pi}\int_V n_1(\bar{r}') \cdot e^{-jk|x-x'|}\,\frac{e^{jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\,dV' \tag{5.17}$$

The above solution can be considerably simplified in the case for which $\lambda << \ell_o$:

Surely, in the above case the angles of scattering of the incident wave by the nonhomogeneities of the refraction index are of the order /12 $\Theta_o = \frac{\lambda}{\ell_o}$ (very small). Therefore, the value ψ_1 ($\bar{r}$) will score substantial variations from point to point, only inside a cone of angle $\Theta_o = \frac{\lambda}{\ell_o} << 1$ and axes coinciding with the axis x and which has its base facing the point of transmission and its vertex at the point of reception.

Inside this volume:

$$|x-x'| >> \sqrt{(y-y')^2+(z-z')^2}$$

but: $|\bar{r}-\bar{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} =$

$$= (x-x')\sqrt{1+\frac{(y-y')^2+(z-z')^2}{(x-x')^2}} \cong$$

$$\cong (x-x')\left[1+\frac{(y-y')^2+(z-z')^2}{2(x-x')^2}\right] = (x-x') + \frac{(y-y')^2+(z-z')^2}{2(x-x')}$$

replacing this value in the expression

$$exp\left[jk\left|\bar{r}-\bar{r}'\right|\right]$$

and considering in the denominator $|\bar{r} - \bar{r}'|$ only the first term of the above expansion, we have

$$\psi_1(\bar{r}) \cong \frac{k^2}{2\pi}\int_V n_1(\bar{r}')\,\frac{exp\left[jk\frac{(y-y')^2+(z-z')^2}{2(x-x')}\right]}{(x-x')}\,dV' \tag{5.18}$$

It can be proven easily that relationship (5.18) is the exact solution of the equation:

$$\frac{\partial^2 \psi_1}{\partial y^2} + \frac{\partial^2 \psi_1}{\partial z^2} + 2jk\frac{\partial \psi_1}{\partial x} + 2k^2 n_1(r) = 0 \tag{5.19}$$

which results from Eq. (5.11) by omitting the term $\frac{\partial^2 \psi 1}{a x^2}$.

Referring to the error which is being introduced in the phase of Eq. (5.16) by omitting the first two terms of the expansion:

$$k\,|\bar{r} - \bar{r}'| = k(x - x') + \frac{k\mu^2}{2(x-x')} + \frac{k\mu^4}{8(x-x')^3} + \ldots,$$

where $\mu - \theta . L \cong \frac{\lambda L}{l_0}$ and $(x - x') \cong L$

we see that the difference is of the order of $\frac{k\mu^4}{(x-x')^2}$. The error is, therefore, negligible while:

$L << \frac{l_0^4}{\lambda^3}$, but because $\lambda << l_0$, $\frac{l_0^4}{\lambda^3} >> \frac{l_0^2}{\lambda}$

and since: $L >> \frac{l_0^2}{\lambda}$ it must ensue that:

$$\frac{l_0^2}{\lambda} << L << \frac{l_0^4}{\lambda^3}.$$

For the case, e.g. where $\ell_0 = 5$ m and $\lambda = 0.1$ m we must have $L << 625$ km which is practically the case for all terrestial radioelectric couplings through troposphere.

3. Solution of Equation (5.19)

We begin by expressing the distribution $n_1(\bar{r}) = n_1(x,y,z)$ with the expansion on the plane x = const.

$$n_1(x,y,z) = n_1(x,o,o) + \int\int_{-\infty}^{+\infty} \left[1 - exp\left\{j(u_2 y + u_3 z)\right\}\right] d\nu(u_2, u_3, x)\,(*) \tag{5.20}$$

*$\nu(u_1, u_2, x)$ is the function of spectral distribution of $n_1(x,y,z)$ on the plane x = const. For further analysis see below and Appendix.

We will search for a solution of Eq. (5.19) of the form:

$$\psi_1(\bar{r}) = \psi_1(x,o,o) + \int\int_{-\infty}^{+\infty} \left[1 - exp\left\{j(u_2 y + u_3 z)\right\}\right] d\varphi(u_2, u_3, x)\,(*) \tag{5.21}$$

Substituting expansions (5.20) and (5.21) in Eq. (5.19), expressing $u_2^2 + u_3^2 = u^2$ and applying $y = z = 0$, we obtain:

$$\int\int_{-\infty}^{+\infty} u^2 . d\varphi(u_2, u_3\ x) + 2jk\frac{d\psi_1(x,o,o)}{dx} + 2k^2 . n_1(x,o,o) = 0 \tag{5.22}$$

Substituting the above expression in expansion (5.19) for $y \neq 0,\ z \neq 0$ we have:

$$\int\int_{-\infty}^{+\infty} -u^2\left[1-e^{j(u_2y+u_3z)}\right] d\varphi(u_2,u_3,x) + 2jk\int\int_{-\infty}^{\infty}\left[1-e^{j(u_2y+u_3z)}\right]\frac{\partial}{\partial x}d\varphi(u_2,u_3,x) + 2k^2\int\int_{-\infty}^{\infty}\left[1-e^{j(u_2y+u_3z)}\right] d\nu(u_2,u_3,x) = 0 \tag{5.23}$$

From Eq. (5.23) it is concluded that the amplitude $d\nu\,(u_2,u_3,x)$ and $d\phi(u_2,u_3,x)$ are related by the relationship:

$$2jk\frac{\partial}{\partial x} d\varphi(u_2, u_3, x) - u^2 . d\varphi(u_2, u_3, x) + 2k^2 d\nu\, u_2,u_3,x) = 0 \tag{5.24}$$

The solution of the above equation which approaches zero for $x = 0$ (which, of course, means that the fluctuations of the field disappear on the boundary surface of the volume inside which the distribution $n_1(x,y,z)$ exists, has /13 the form:

* $(u_3,\ u_3\ x)$ is the spectral distribution function of $\psi_1(x,y,z)$ on the plane x = const. The spectral component u must not be confused with the expression of the electric field noted in paragraphs II 1 and II 2.

$$d\varphi(u_2, u_3, x) =$$

$$= jk\int_0^x dx' \exp\left[-\frac{ju^2(x-x')}{2k}\right] d\nu(u_2, u_3, x') \tag{5.25}$$

From relation (5.21) we have:

$$x(\bar{r}) = Re\left[\psi_1(x,o,o)\right] +$$

$$+ Re\left[\int\int_{-\infty}^{\infty}\left[1-e^{j(u_2y+u_3z)}\right] d\varphi(u_2,u_3,x) =\right.$$

$$= Re\left[\psi_1(x,o,o)\right] + \frac{1}{2}\int\int_{-\infty}^{\infty}\left[1-e^{j(u_2y+u_3z)}\right] d\varphi(u_2 u_3 x) +$$

$$+ \frac{1}{2}\int\int_{-\infty}^{\infty}\left[1-e^{-j(u_2y+u_3z)}\right] d\varphi^*(u_2,u_3,x)$$

or $x(\bar{r}) = Re\left[\psi_1(x,o,o)\right] +$

$$+ \int\int_{-\infty}^{\infty}\left[1-e^{j(u_2y+u_3z)}\right] \frac{d\varphi(u_2u_3x) + d\varphi^*(-u_2-u_3,x)}{2} \tag{5.26}$$

Similarly we find: $\varphi_1 = Im\{\psi_1\}$ & therefore*

$$\varphi_1(\bar{r}) = Im\left[\psi_1(x,o,o)\right] +$$

$$+ \int\int_{-\infty}^{\infty}\left[1-e^{j(u_2y+u_3z)}\right] \frac{d\varphi(u_2,u_3,x) - d\varphi^*(-u_2-u_3,x)}{2j} \tag{5.27}$$

We designate:

$$da(u_2,u_3,x) = \frac{d\varphi(u_2,u_3,x) + d\varphi^*(-u_2,-u_3,x)}{2}$$

and $$d\sigma(u_2,u_3,x) = \frac{d\varphi(u_2,u_3,x) - d\varphi^*(-u_2,-u_3,x)}{2j}$$

Substituting in Eq. (5.25) we obtain:

$$da(u_2,u_3,x) = k\int_0^x dx' \sin\left[\frac{u^2(x-x')}{2k}\right] d\nu(u_2u_3,x') \tag{5.28}$$

$$d\sigma(u_2,u_3,x) = k\int_0^x dx' \cos\left[\frac{u^2(x-x')}{2k}\right] d\nu(u_2,u_3,x)' \tag{5.29}$$

*The expression of the phase ϕ_1 should not be confused with the differential $d\phi$.

Let us make a physics analysis of relations (5.28) and (5.29):

The expressions da and $d\sigma$ (see Eqs. 5.26 and 5.27) represent the respective spectral width components of the fluctuations of the logarithm of the amplitude $x(\bar{r})$ and phase $\phi_1(\bar{r})$ of the scattered electromagnetic wave.

We observe that nonhomogeneities of the refraction index of dimension $\ell = \frac{2\pi}{u}$ which are at a distance x-x' from the point of observation (receiver), appear in the determination of the amplitude da and $d\sigma$ with weighted quantities $\sin \frac{\pi\Lambda^2}{\ell^2}$ and $\cos \frac{\pi\Lambda^2}{\ell^2}$ respectively where $\Lambda^2 = \lambda|x-x'|$ is the square of the radius corresponding to the first Fresnel belt.

The degree of influence of a nonhomogeneity of the refraction coefficient on the fluctuations $x(\bar{r})$ and $\phi_1(\bar{r})$ depends on the relationship between the dimensions of the discontinuity ℓ under consideration and the radius of the first Fresnel belt in the region of the nonhomogeneity ℓ. This degree of influence, as long as the frequency of the propagated electromagnetic oscillation increases, decreases for the amplitude fluctuations and increases for the phase fluctuations.

Multiplying the expression (5.28) with its conjugate:

$$da^*(u_2', u_3', x) = k\int_0^x dx'' \sin\left[\frac{u'^2(x-x'')}{2k}\right] . d\nu^*(u_2', u_3', x'')$$

and taking the average value of the product, we have:

$$\overline{da(u_2, u_3, x) . da^*(u_2', u_3' x)} = k^2 \int_0^x dx' \int_0^x dx'' \sin\left[\frac{u^2(x-x')}{2k}\right] \sin\left[\frac{u'^2(x-x'')}{2k}\right] . \overline{d\nu(u_2, u_3, x') . d\nu^*(u_2', u_3', x'')}$$

It is easy to prove that a locally nonhomogeneous field $f(\bar{r})$ developing at the origin can be written

$$f(\bar{r}) = f(o) + \iiint_{-\infty}^{\infty} \left(1 - e^{j\bar{u}\bar{r}}\right) d\nu(\bar{u}) \qquad \text{(A)}$$

The spectral width $dv(\underline{u})$ of the above field satisfies the relationship:

$$\overline{dv(\bar{u}_1) \cdot dv^*(\bar{u}_2)} = \delta(\bar{u}_1 - \bar{u}_2) \cdot \phi(\bar{u}_1) \cdot d\bar{u}_1 \, d\bar{u}_2 \quad \text{(B)}$$

where $\phi(\bar{u}) \geq 0$ is the spectral density of the function $f(\bar{r})$ and δ the Delta function of Dirac.

Accordingly we have:

$$\overline{d\nu(u_2, u_3, x)\, d\nu^*(u_2', u_3', x'')} = \delta(u_2 - u_2') \cdot \delta(u_3 - u_3') \cdot F_\nu(u_2, u_3, x' - x'')\, du_2 \cdot du_3 \cdot du_2' \cdot du_3' \quad \text{(5.31)}$$

and similarly:

$$\overline{da(u_2, u_3, x)\, da^*(u_2', u_3', x)} = \delta(u_2 - u_2')\, \delta(u_3 - u_3')\, F_a(u_2, u_3, 0)\, du_2 du_3\, du_2' du_3' \quad \text{(5.32)}$$

where: $F_v(u_2, u_3, x'-x'')$ is the two-dimensional) function of the refraction index spectral distribution and $F_a(u_2, u_3, 0)$ the (two-dimensional) function of the spectral distribution of the structure function on the variations x on the plane x = const.

Let us prove this by more detailed analysis of the meaning of the above function: /14

The "structure"* function of the "random" and "stationary" function $f(\bar{r})$ represents the strength of the fluctuations of it for distances $r \leq r_1$ and is given as

$$D(\bar{r}) = \overline{[f(\bar{r}+\bar{r}_1) - f(\bar{r}_1)]^2} \quad \text{(C)}$$

and $D(r) = 2\,[B(o) - B(r)]$ where $B(r) = \overline{f(r)\, f(r+r)}$ the correlation function of $f(r)$.

Considering the relations (A), (B), and (C) we have:

$$D(\bar{r}) = 2 \iiint_{-\infty}^{\infty} (1 - \cos \bar{u}\bar{r}) \cdot \Phi(\bar{u}) \cdot d\bar{u} \quad \text{(D)}$$

If the field is also locally isotropic, i.e. that inside the volume ΔV of its isotropic character depends only on the measure of $|\bar{r}| = r_1$ then:

$$D(r) = 8\pi \int_{-\infty}^{\infty} \left(1 - \frac{\sin ur}{u.r}\right) \cdot \Phi(u) \cdot du \quad \text{(E)}$$

*Relative to the meaning of the structure function see attached Appendix.

Let us now examine the case of a locally isotropic "random" field developing on two dimensions on the plane x = const. (for example).

Is:

$$f(x, y, z) = f(x, o, o) + \int\int_{-\infty}^{\infty}\left\{1-exp\left[j(u_2 y + u_3 z)\right]\right\} d\psi(u_2, u_3, x)$$

where the function ψ gives the spectral width of f(x,y,z) and satisfies the condition:

$$\overline{d\psi(u_2, u_3, x) \cdot d\psi^*(u_2', u_3', x')} = \delta(u_2 - u_2') \cdot \delta(u_3 - u_3') \cdot F(u_2, u_3, |x - x'|) \, du_2 \, du_3 \, du_2' \, du_3' \tag{F}$$

where $F(u_2, u_3, x - x')$ is the two-dimensional function of the spectral distribution of the field f(x,y,z) between two points on the plane x = const, we have:

$$f(x, y, z) - f(x, y', z') = \int\int_{-\infty}^{\infty}\left\{exp\left[j(u_2 y' + u_3 z')\right] - exp\left[j(u_2 y + u_3 z)\right]\right\} \cdot d\psi(u_2, u_3, x)$$

If we now calculate the correlation function of two such differences taken on the planes x and x', we have:

$$\overline{[f(x, y, z - f(x, y', z')] \cdot [f(x', y, z) - f(x', y', z')]} = \int\int\int\int_{-\infty}^{\infty}\left\{exp\left[j(u_2 y' + u_3 z')\right] - exp\left[j(u_2 y + u_3 z)\right]\right\} \cdot \left\{exp\left[-j(u_2' y' + u_3' z')\right] - exp\left[-j(u_2' + u_3' z)\right]\right\} \cdot \overline{d\psi(u_2, u_3, x) \, d\psi^*(u_2', u_3', x)} \tag{Z}$$

Using Eq. (F), the right-hand side of Eq. (G) becomes:

$$2\int\int_{-\infty}^{\infty}\left\{1 - ovr\left\{u_2(y - y') + u_3(z - z')\right\}\right\} F(u_2, u_3, |x - x'|) \, du_2 \, du_3 \tag{H}$$

Essentially, a correlation between the differences

$$\{f(x,y,z) - f(x,y',z')\} \quad \text{and} \quad \{f(x',y,z) - f(x'y'z')\}$$

is created only by nonhomogeneities, the dimensions of which are greater than the distance $|x-x'|$ of the two planes x and x', i.e. for distances $\ell \geq |x-x'|$ or for $u\,|x-x'| \leq|$ $(u = 2\pi/\ell)$.

As a result, the function $F(u_2, u_3\,|x-x'|)$ which is the spectral distribution function of the expression

$$\overline{[f(x,y,z) - f(x,y',z')] \cdot [f(x',y,z) - f(x'y'z')]}$$

decreases quickly for

$$u\,|x-x'| > 1$$

Using the identity

$$(\alpha-\beta)(\gamma-\delta) \equiv \frac{1}{2}\left[(\alpha-\delta)^2+(\beta-\gamma)^2-(\alpha-\gamma)^2-(\beta-\delta)^2\right]$$

we can express Eq. (H) in terms of the structure function of the field f:

$$D(x-x', y-y', z-z') - D(x-x', o, o) =$$
$$=2\int\int_{-\infty}^{\infty}\left\{1- (u_2([y-y']+u_3[z-z'])\right\} \cdot F(u_2, u_3, |x-x'|)\,du_2\,du_3 \tag{I}$$

Applying $x=x'$, $y-y'=\gamma$ and $z-z'=\xi$ we obtain:

$$D(o, \gamma, \xi) = 2\int\int_{-\infty}^{\infty}\left[1- (u_2\gamma+u_3\xi)\right] \cdot F(u_2, u_3, 0)\,du_2 \cdot du_3 \tag{J}$$

i.e. as was stated and for $F_a(u_2, u_3, o)$ above, the $F(u_2, u_3, o)$ is the two-dimensional spectral distribution function of the structure function $D(n, \xi)$ of the field $f(\bar{r})$, as we were to prove.

In the case of local isotropic conditions on the plane x = const., $F(u_2, u_3, |x|)$ depends only on the parameter $u = \sqrt{u_2^2 + u_3^2}$ and, therefore, considering the relation

$$\int_0^{2\pi} (x\cos v\tau)\,d\tau = 2\pi J_0(x)$$

where $J_0(x)$ is the Bessel function of zero order we have: /15

$$D(\varrho) = 4\pi \int_0^\infty \left[1 - J_0(u\varrho)\right] F(u, o)\, u\, du \tag{K}$$

where $\varrho^2 = \gamma^2 + \xi^2$ & $F(u_2, u_3, o) = F\left(\sqrt{u_2^2 + u_3^2}, o\right)$

In the case where $f(\bar{r})$ is homogeneous and isotropic on the plane x = const.:

$$B(\varrho) = 2\pi \int_0^\infty J_0(u, \varrho)\, F(u . o), u . du \tag{L}$$

It remains now only to express the relation between the two-dimensional function of the spectral distribution of $D(\rho)$, $F(u_2, u_3, x)$ and the three-dimensional function of the spectral distribution of $D(r)$, $\phi(u_1, u_2, u_3)$ [see Eq. (D)]. Substituting from Eq. (D) into the left member of Eq. (I), and considering that $\phi(u_1, u_2, u_3)$ is an even function, we obtain:

$$F(u_2, u_3, x) = \int_{-\infty}^{\infty} \cos(u, x) . \Phi(u_1, u_2, u_3)\, du$$

or $$\Phi(u_1, u_2, u_3) = \frac{1}{2\pi} . \int_{-\infty}^{\infty} F(u_2, u_3\, x) . \cos(u, x) . dx$$

and for the plane $x = o$:

$$\Phi(o, u_2, u_3) = \frac{1}{\pi} \int_0^\infty F(u_2, u_3, \xi)\, d\xi \tag{N}$$

Let us return now, after the above explanations [Eq. (A) up to (N)], to Eqs. (5.31) and (5.32):

Substituting the above in Eq. (5.30) we obtain:

$$F_a(u_2, u_3, o) = k^2 \int_0^x \int_0^x \left(\frac{u^2(x-x')}{2k}\right) . \sin\left(\frac{u^2(x-x'')}{2k}\right) F_v(u_2, u_3, |x'-x''|)\, dx'dx'' \tag{5.33}$$

Similarly from Eq. (5.29) we achieve:

$$F_\varphi(u_2, u_3, o) = k^2 \int_0^x \int_0^x \cos\left(\frac{u^2(x-x')}{2k}\right) . \cos\left(\frac{u^2(x-x'')}{2k}\right) F_v(u_2, u_3, |x'-x'')\, dx'dx'' \tag{5.34}$$

The expressions (5.33) and (5.34) can be substantially simplified:

Primarily, $F_v(u_2, u_3, x'-x'') = F_v(u_2, u_3, x''-x')$ allows the introduction into Eqs. (5.33) and (5.34) of the variables of integration:

$$\xi = x'-x'' \quad \text{and} \quad \lambda = \frac{x' + x''}{2}$$

The integration with respect to γ can easily be achieved because F_v is independent of γ. Designating the abscissa of the point of observation (receiver) $x = L$, we obtain

$$= \int_0^L \left\{ k^2 (L-\xi) . \cos\left(\frac{u^2 . \xi}{2k}\right) + \frac{k^2}{u^2} \sin\left(\frac{u^2 \xi}{2k}\right) - \frac{k^2}{u^2} . \sin\left(\frac{u^2(2L-\xi)}{2k}\right) \right\} . F_v(u_2, u_3, \xi)\, d\xi \tag{5.35}$$

and $F_\varphi(u_2, u_3, o) =$

$$= \int_0^L \left\{ k^2 (L-\xi) \cos\frac{u^2\xi}{2k} - \frac{k^2}{u^2} \sin\frac{u^2\xi}{2k} + \frac{k^2}{u^2} \sin\frac{u^2(2L-\xi)}{2k} \right\} F_v(u_2, u_3, \xi)\, d\xi \tag{5.36}$$

As it was referred to above, the function F_v decreases rapidly towards zero for $u \cdot \xi > 1$. Therefore, the substantial contribution of F_v to the above two integrals occurs for $\xi < 1/u$.

In this region the contribution is $\frac{u^2\xi}{k} \leq \frac{u}{k}$. On the other hand, we have assumed that $\lambda \ll \ell_o$; however, $\ell_o \sim \frac{1}{u_m}$ where u_m is the maximum value of the parameter u, for which $F_v(u,\xi) \neq 0$. Therefore, we have $\frac{1}{k} \ll \frac{1}{u_m}$ and $\frac{u}{k} < \frac{u_m}{k} \ll 1$. Therefore, the relation $\frac{u^2\xi}{2k} \ll 1$ is fulfilled inside the essential area of integration and we can state:

$$\cos\frac{u^2\xi}{2k} \approx 1, \sin\frac{u^2\xi}{2k} \approx \frac{u^2\xi}{2k} . \frac{\sin[u^2(2L-\xi)]}{2k} \approx \sin\frac{u^2 . L}{k} .$$

We will calculate the structure function (or the correlation function) of the functions x and ϕ from Eqs. (5.35) and (5.36) for the values of turbulence scale $\ell \ll L$, i.e. $\frac{1}{u} \ll L$.

Because the essential area of integration extends to the values $\xi \ll \frac{1}{u}$, inside this region we will have $\xi \ll L$. Considering then all the above simplifying conditions we have:

$$F_a(u_2, u_3, o) = \int_0^L \left(k^2 . L - \frac{k^2}{u^2} \sin\frac{u^2 L}{k}\right) . F_v(u_2, u_3, \xi)\, d\xi \tag{5.37}$$

and
$$F_\varphi(u_2, u_3, o) = \int_0^L \left(k^2 . L + \frac{k^2}{u^2} \sin\frac{u^2 L}{k}\right) . F_v(u_2, u_3, \xi)\, .\, d\xi \tag{5.38}$$

Because the function F_v (u_1, u_2, ξ) decreases rapidly to zero for large values ξ, the above integrals (5.37) and (5.38) can be extended from zero to infinity without noticeable change in the values of the integrals.

And since (see Eq. (N))

$$\int_0^\infty F_v(u_2, u_3, \xi)\, d\xi = \pi\Phi(o, u_2, u_3)$$ /16

where Φ $(\bar{u})$ represents the (three-dimensional) spectral distribution of the structure function of the discontinuities of the refraction index, we have:

$$F_a(u_2, u_3, o) = \\ = \pi k^2 . L . \left(1 - \frac{k}{u^2 . L} . \sin\frac{u^2 L}{k}\right) . \Phi(o, u_2, u_3) \tag{5.39}$$

$$F_\varphi(u_2, u_3, o) = \\ = \pi k^2 . L . \left(1 + \frac{k}{u^2 . L} . \sin\frac{u^2 . L}{k}\right) . \Phi(o, u_2, u_3) \tag{5.40}$$

In the present example where the field of the refraction index is considered locally isotropic, we have:

$$\phi(u_1, u_2, u_3) = \phi\left(\sqrt{u_1^8 + u_2^2 + u_3^8}\right)$$

Therefore: designating $u^2 = u_2^2 + u_3^2$ ·

$$F_\alpha(u_1, u_2, o) = F_\alpha(u, o)$$

$$F_\varphi(u_2, u_3, o) = F_\varphi(u, o) \; , \qquad \Phi(o, u_2, u_3) = \Phi(u)$$

we have

$$F_\alpha(u, o) = \pi k^2 . L \left(1 - \frac{k}{u^2 L} \sin \frac{u^2 L}{k}\right) \Phi(u) \tag{5.41}$$

and

$$F_\varphi(u, o) = \pi k^2 . L \left(1 + \frac{k}{u^2 L} \sin \frac{u^2 L}{k}\right) \Phi(u) \tag{5.42}$$

The above expressions relate the (two-dimensional) functions of spectral distribution of the structure functions of the amplitude and phase fluctuations of the electromagnetic field, on the plane x = const. = L to the (three-dimenional) functions of spectral distributions of the structure function of the refraction index.

The structure functions, therefore, of the distributions x and ϕ to be found at the place of reception (x = L), will result from the relations (see Eq. (K)):

$$D_\alpha(\varrho) = \overline{[x(L, y, z) - x(L, y', z')]^2} = 4\pi \int_0^\infty \left[1 - J_0(u\varrho)\right] F_\alpha(u, o)\, u\, du \tag{5.43}$$

and:

$$D_{\varphi_1}(\varrho) = \overline{[\varphi_1(L, x, y) - \varphi_1(L, x', y')]^2} = 4\pi \int_0^\infty \left[1 - J_0(u\varrho)\right] F_\varphi(u, o)\, u\, du \tag{5.44}$$

where: $\varrho^2 = (y - y')^2 + (z - z')^2$

The correlation functions of the distributions x and ϕ will be respectively: (under the condition that the field of the refraction index is homogeneous and isotropic for all scales of turbulence).

$$B_\alpha(\varrho) = \overline{x(L, y, z) \cdot x(L, y', z')} = 2\pi \int_0^\infty J_0(u\varrho)\, F_\alpha(u, o)\, u\, du \tag{5.45}$$

and:

$$B_{\varphi_1}(\varrho) = \overline{\varphi_1(L, y, z)\,\varphi_1(L, y', z')} = $$
$$= 2\pi \int_0^\infty J_0(u\varrho)\, F_\varphi(u, o)\, u\, du \qquad (5.46)$$

Replacing the values of $F_a(u,o)$ and $F\phi(u,o)$ from (5.41) and (5.42) we obtain:

$$B_\alpha(\varrho) = \overline{x(L, y, z)\,.\,x(L, y', z')} = $$
$$= 2\pi^2 k^2 L \int_0^\infty J_0(u\varrho)\left(1 - \frac{k}{u^2 L}\sin\frac{u^2 L}{k}\right).\ \Phi(u)\, u du$$

and $B_\varphi(\varrho) = \overline{\varphi_1(L, y, z)\,.\,\varphi_1(L, y' z')} =$

$$= 2\pi^2 k^2 L \int_0^\infty J_0(u\varrho)\left(1 + \frac{k}{u^2 L}\sin\frac{u^2 L}{k}\right).\ \Phi(u) u du$$

The mean square deviations of the fluctuations $\overline{x^2}$ and $\overline{\phi_1^{\,2}}$ to be found will result as:

$$u^2 = B_\alpha(o) = $$
$$= 2\pi^2 k^2 L \int_0^\infty \left(1 - \frac{k}{u^2 L}\sin\frac{u^2 L}{k}\right)\Phi(u)\, u.du \quad [J_0(o) = 1]$$

and $\overline{\varphi_1^2} = B_\varphi(o) =$

$$= 2\pi^2 k^2 L \int_0^\infty \left(1 + \frac{k}{u^2 L}\sin\frac{u^2 L}{k}\right)\Phi(u)\, u.du \qquad (5.43)$$

As is known (see Part A.), the function $\phi(u)$ of spectral distribution of the fluctuations of the refraction index determines also the equivalent scattering cross section of the scattering volume V inside the solid angle $d\Omega$, i.e.:

$$d\sigma(\Theta) = 2\pi^2 .\, k^4\, V,\ \Phi\left(2k\sin\frac{\Theta}{2}\right) d\Omega \qquad (5.44)$$

or per unit volume

$$d\sigma_0(\Theta) = 2\pi^2 .\, k^4 .\ \Phi\left(2k\sin\frac{\Theta}{2}\right) .\ d\Omega \qquad (5.45)$$

We now apply, in the integrals of Eq. (5.43), the change of variable

$u = 2k\sin\frac{\Theta}{2}$.

The variable u in Eq. (5.43) oscillates from zero to infinity.

The variable $2k \sin \frac{\Theta}{2}$ takes respectively the values from 0 ...... 2k (for real Θ). The value of $\phi(u)$ is almost zero for $u >> u_m$. Therefore, in the example under consideration $u >> u_m$, the upper limit of integration in Eq. (5.43) can be substituted by a number much greater than u_m, e.g. $\frac{2k}{\sqrt{2}}$ corresponding to $\Theta = \frac{\pi}{2}$. Substituting we obtain (for the amplitude fluctuations, e.g.)

$$\overline{x^2} = 2\pi^2 k^2 L \int_0^{\pi/2} \left[1 - \frac{\sin\left(4kL\sin^2\frac{\Theta}{2}\right)}{4kL\sin^2\frac{\Theta}{2}}\right] \cdot$$

/17

$$\cdot\, \Phi\left(2k\sin\frac{\Theta}{2}\right) k^2 \sin\Theta\, d\Theta$$

or considering Eq. (5.45):

$$\overline{x^2} = \pi L \int_0^{\pi/2} \left[1 - \frac{\eta\mu\left(4k\,L\sin^2\frac{\Theta}{2}\right)}{4kL\sin^2\left(\frac{\Theta}{2}\right)}\right] \frac{d\sigma_0(\Theta)}{d\Omega} \sin\Theta . d\Theta$$

The quantity $\frac{d\sigma_0(\Theta)}{d\Omega}$ represents the equivalent scattering cross section in an angle Θ inside the solid angle $d\Omega = \sin\Theta \,.\, d\Theta \,.\, d\phi$.

If we define $d\sigma_1(\Theta)$ as the equivalent scattering cross section inside the solid angle $d\Omega = 2\sin\Theta d\Theta$ which is limited by the cones of angle Θ and $\Theta + d\Theta$, this will be given by

$$\frac{d\sigma_0}{d\varphi} = \frac{d\sigma_1}{2\pi} \text{ and } \frac{d\sigma_0}{\Theta d\Theta d\varphi} \Theta d\Theta = \frac{d\sigma_1}{2\pi}$$

Therefore

$$\overline{x^2} = \frac{1}{2} L \int_0^{\pi/2} \left[1 - \frac{\sin\left(4kL\sin^2\frac{\Theta}{2}\right)}{4kL\sin^2\frac{\Theta}{2}}\right] d\sigma_1(\Theta) \qquad (5.46)$$

As we saw in Part A (Eq. 16'), the dimension of the atmospheric nonhomogeneities ("clots") which create scattering in an angle Θ equals to

$\ell(\Theta) = \frac{\lambda}{2\sin\frac{\Theta}{2}}$. The quantity then $4kL\sin^2\frac{\Theta}{2} = \frac{2\pi}{\lambda} L \frac{\lambda^2}{\ell^2(\Theta)} = 2\,\frac{\lambda L}{\ell^2(\Theta)}$ is proportional to the square of the ratio of the radius of the first Fresnal belt to the diameter $\ell(\Theta)$ of the atmospheric discontinuities which cause scattering in an angle Θ.

The function of $1-\frac{\sin\left(\frac{2\pi\lambda L}{l^2(\Theta)}\right)}{\frac{2\pi\lambda L}{l^2(\Theta)}}$ gives a maximum for $\ell(\Theta) \sim \sqrt{\lambda L}$. So, as it is seen from Eq. (5.46), the fluctuations of the logarithm of the amplitude are created by superposition of the scattered components $d\sigma_1(\Theta)$ with different weighed factors [see expression inside the brackets of Eq. (5.46)]. The larger weighted factor, as we indicate above, is given by the scattered components which come from tropospheric clots of dimension comparable to the radius of the first Fresnel belt.

In the case where the nonhomogeneities $\ell_1(\Theta)$ are large compared to $\sqrt{\lambda L}$, they act not as "diffusers" but as "coherent" sources of partial reflections.

In this case:

$$1-\frac{l^2(\Theta)}{2\pi\lambda L}\sin\left(\frac{2\pi\lambda L}{l^2(\Theta)}\right) \cong \frac{1}{6}\, 16\, k^2 L^2 \sin^4\frac{\Theta}{2}$$

then

$$\overline{x^2} = \frac{4}{3} k^2 L^3 \int_0^{\pi/2} \sin^4\frac{\Theta}{2}\cdot d\sigma_1(\Theta) \tag{5.47}$$

The case (5.47) corresponds to the geometric optics.

In the inverse case (the radius of first Fresnel belt is much larger than the scale of turbulence) $\frac{\sqrt{\lambda L}}{\ell(\Theta)} \gg 1$, then Eq. (5.46) becomes:

$$\overline{x^2} \cong \frac{1}{2} L \int_0^{\pi/2} d\sigma_1(\Theta) = \frac{1}{2}\sigma\cdot L \tag{5.48}$$

where σ represents the equivalent scattering cross section of unit common volume.

This parameter determines, as it is seen, the per unit length reduction of the electromagnetic wave due to scattering.

Let us now return to the relation (5.43) and find another parameter (besides σ which was examined above) which could be measured experimentally for expressing x^2 and ϕ_1^2.

For $\sqrt{\lambda L} \gg L_o$ where L_o is the maximum scale of turbulence

$$\left(1 \pm \frac{k}{u^2 L}\sin\frac{u^2 L}{k}\right)\Phi(u) \cong \Phi(u)$$

but 'Αλλά $$\Phi(u) = \frac{1}{2\pi^2 u}\int_0^\infty B(r)\sin(ur).r.dr$$

where B(r) is the correlation function of the fluctuations of the refraction index.

(We are reminded that the spectral distribution of the fluctuations of a "random" function and the correlation function of the above fluctuations are related by the Fourier transformation.)

From the above ensues:

$$\overline{x^2} = \overline{\varphi_1^2} = k^2.L\int_0^\infty B(r).r.dr \int_0^\infty \sin(ur)\,du$$

but $\int_0^\infty \sin(ur)\,du = \frac{1}{r}$ then

$$\overline{x^2} = \overline{\varphi_1^2} = k^2.L\int_0^\infty B(r).dr \tag{5.49}$$

As an example:

Assume a turbulent field statistically homogeneous and isotropic which is described by the correlation function of the fluctuations of the refraction index:

$$B(r) = \overline{n_1^2}\, e^{-r^2/a^2} \qquad \text{(Gauss' field)*}$$

where a is the (internal or external in this case) scale of turbulence.

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*Systematic experimental verifications (see, e.g., J. Grosskopf "Fading Investigations for Tropospheric Propagation Paths" Proceedings of Commission II during the XIII General Assembly of URSI, London, 1960) show that the correlation function of the fluctuations of the refraction index is now correctly written as $B(r) \sim \exp\left(-\frac{r}{a}\right)^n$. For small values r we have:

$$B(r) \backsim 1 - \left(\frac{r}{a}\right)^n, \quad \log\left\{1 - B(r)\right\} = n\log r - n\log a$$

$$\text{then } \frac{d(\log[1 - B(r)])}{d(\log r)} = n \text{ and } a = r\Big/\sqrt[n]{1 - B(r)}$$

The determination of B(r) is achieved experimentally by determining the value r_o for which $B(r_o) = 0.9$, the above relations give the value n (depending as experimentally proven on the frequency and varying between 1.3 ~ 2.3) and the corresponding value a of the scale of turbulence.

The corresponding function of spectral distribution of the scale of turbulence $\ell_1 = 2\pi/u_1$ will be: /18

$$\Phi(u) = \frac{n_1^2 \cdot a^3}{8\pi\sqrt{\pi}} \cdot e^{-\frac{u^2a^2}{4}}$$

The above conditions are sufficient for the determination of $\overline{x^2}$ and $\overline{\phi_1^2}$.

In fact, (see Eqs. 5.41, 5.42)

$$F_\alpha(u,o) = \frac{1}{8\sqrt{\pi}} \overline{n_1^2}\, a^3 k^2 L \left(1 - \frac{k}{u^2L} \sin\frac{u^2L}{k}\right) exp\left(-\frac{u^2a^2}{4}\right)$$

and

$$F_\varphi(u,o) = \frac{1}{8\sqrt{\pi}} \overline{n_1^2}\, a^3 k^2 L \left(1 + \frac{k}{u^2L} \sin\frac{u^2L}{k}\right) exp\left(-\frac{u^2a^2}{4}\right)$$

then: $\left\{ B_\alpha(\varrho) = 2\pi \int_0^\infty J_0(u,\varrho)\, F_\alpha(u,o)\, u\,du \right\}$ (see 5.43, 5.44, 5.45)

$$\overline{x^2} = 2\pi \int_0^\infty F_\alpha(u,o)\, u\,du =$$

$$= \frac{\sqrt{\pi}}{4} \overline{n_1^2}\, a^2 k^2 L \int_0^\infty \left(1 - \frac{k}{u^2L}\sin\frac{u^2L}{k}\right) e^{-\frac{u^2a^2}{4}}\, u\,du$$

and

$$\boxed{\begin{aligned} \overline{x^2} &= \frac{\sqrt{\pi}}{2} \overline{n_1^2}\, a.k^2L \left(1 - \frac{\tau o\xi\, \varepsilon\varphi\, \frac{4L}{ka^2}}{\frac{4L}{ka^2}}\right) \\ \overline{\varphi_1^2} &= \frac{\sqrt{\pi}}{2} \overline{n_1^2}\, a.k^2L \left(1 + \frac{\tau o\xi\, \varepsilon\varphi\, \frac{4L}{ka^2}}{\frac{4L}{ka^2}}\right). \end{aligned}}$$

Let us now examine two basic examples depending on whether the expression

$$T = \frac{4L}{ka^2} \leq 1 \quad \text{or} \quad T > 1$$

Example 1) $\quad T \ll 1$ or $L \ll \frac{\pi a^2}{2\lambda} = L_{(critical)}$

In this case $\frac{1}{T}\tan^{-1} T \simeq 1 - \frac{T^3}{3}$ and $\overline{x^2} = \frac{8\sqrt{\pi}}{3}\, \overline{n_1^2}\, \frac{L^8}{a^8}$ (independent of frequency).

$$\overline{\phi_1^2} = \sqrt{\pi}\, \overline{n_1^2}\, ak^2 \;.\; L$$

(These results correspond to the case of geometric optics).

Example 2) If $T \gg 1$ $L \gg L_{(critical)}$ and $\frac{1}{T} \tan^{-1} T \ll 1$ then:

$$\overline{x^2} = \overline{\phi_1^2} = \frac{\sqrt{\pi}}{2} \overline{n_1^2}\, k^2\, a \;.\; L = 2\pi^2 \sqrt{\pi} \;.\; \frac{\overline{n_1^2}}{\lambda^2} \;.\; aL$$

Conclusions: The statistical analysis of the phenomenon of microwave scattering in the troposphere was given above for the geometric optics approximation and for the general case of obtaining solutions of the wave equations with the help of the theory of "small" and "smooth" disturbances by Kolmogorov. In the first part the elementary equivalent, scattering cross section in an angle Θ was calculated and it was proven that this angle acts as a "filter" allowing the superposition of a minute number of spectral components to constitute the received signal in the above angle.

We also attempted to give a physical interpretation of the manner in which the turbulent condition is being created and developed according to the theory of Kolmogorov-Obukhov*. In the second part the problems of amplitude and phase fluctuations of the received signal was examined from the point of view of correlating these, on one hand to the fundamental parameter of the equivalent scattering cross section, and on the other hand to the frequency of radiation, the length of coupling and the spectrum of turbulence.

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*Analysis of this theory follows in the attached Appendix.

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Presented above are only some contributions on the subject developed. For a full bibliography (including about 500 references) the reader is referred to the works of F. Du Castel "Propagation Tropospherique et Faisceaux Hertziens Transhorizons", Edit. Chiron Paris, 1961.

APPENDIX

THE MEANING OF THE ENERGY DISTRIBUTION FUNCTION OF THE SCALE OF TURBULENCE

1. Let us consider, for example, a time function $f(t)$ "random" and not necessarily statistically stationary (which means a function the characteristic components of which do not change after the time the phenomenon recording begins). It is known that stationary functions are described by their autocorrelation function $B(\tau) = \overline{f(t) \cdot f(t + \tau)}$, which gives the fluctuations of $|f(t)|^2$.

In order to describe the nonstationary functions (which are mainly encountered in the turbulence theory) in a somewhat statistically analogous manner, we use instead of $B(\tau)$, the so-called "structure" function of the function $f(t)$ which is expressed as $D(\tau) = \overline{f(t + \tau) - f(t)^2} = \overline{[f(t + \tau)]^2} + \overline{[f(t)]^2} - 2\overline{f(t + \tau)\; f(\tau)} = 2[B(o) - B(\tau)]$, proposed originally by Kolmogorov.

The basical thought for this is that in cases where $f(t)$ is nonstationary, i.e. $f(t)$ changes with time, if we consider $F(t) = f(t + \tau) - f(t)$ for values of t not very large, small changes of $f(t)$ do not influence $F(t)$ which now can be considered as stationary.

Since now $B(\tau)$ constitutes the Fourier integral of the spectral distribution of $W(\omega)$*

$$B(\tau) = \int_{-\infty}^{\infty} e^{j\omega\tau} \cdot W(\omega)\, d\omega = \int_{-\infty}^{\infty} \sigma v\nu(\omega\tau) \cdot W(\omega)\, d\omega$$

we will have:

$$D(\tau) = 2\int_{-\infty}^{\infty} (1 - \sigma v\nu(\omega\tau))\; W(\omega)\, d\omega$$

If, on the other hand, we call $\phi(\omega)$ the spectral density of the spectral components of $f(t)$, i.e.

$$f(t) = \int_{-\infty}^{\infty} e^{j\omega\tau} \cdot \varphi(\omega)\, d\omega$$

we will have:

$$\overline{\varphi(\omega_1)\, \varphi^*(\omega_2)} = \delta(\omega_1 - \omega_2) \cdot W(\omega_1)\, d\omega_1 d\omega_2$$

relation which relates the mean spectral density of the fluctuations energy of $f(t)$ to the spectral density of it. The function $\delta(\omega)$ is known as the Dirac function.

* $W(\omega)$ represents the mean spectral density of the energy (power) of the fluctuations of $f(t)$.

2. Let us come now to the examination of "random" functions of three variables ("random" fields). Examples of such functions are the field of the wind velocity in a turbulence atmosphere (vector), or the humidity fields, temperature etc. (scalar) inside the same medium. For such a field $f(\bar{r})$ we can also specify the correlation function:

$$B(\bar{r}_1 , \bar{r}_2) = \overline{[f(\bar{r}_1) - \overline{f(\bar{r}_1)}] \cdot [f(\bar{r}_2) - \overline{f(\bar{r}_2)}]}$$

If the field is homogeneous, $B(\bar{r}_1, \bar{r}_2)$ is a function only of the difference $\bar{r}_1 - \bar{r}_2$, and if the homogeneous field is also isotropic then

$$B(r_1, r_2) = B(|\bar{r}_1 - \bar{r}_2|).$$

3. We will deal especially with the examination of the field of the wind velocity - which is responsible for the creation of turbulence in the troposphere[1], and, therefore, the field of the refraction index.

This field creates stratification of the turbulence from the (maximum) scale Lo to the (minimum) ℓ_o.

The regions of dimensions L_o which appear as results of the instability of the original flow, are naturally nonisotropic as they depend directly on the geometrical peculiarities of the flow. These conditions though do not affect the clots of high order of turbulence ℓ_i (in the region ℓ_o) and, therefore, we can consider their regions as isotropic. Since the "clots" of dimension

$$l >> |\bar{r}_1 - \bar{r}_2|$$

do not influence the velocity difference

$$\bar{v}(\bar{r}_1) - \bar{v}(\bar{r}_2)$$

we can consider that, for differences $|\bar{r}_1 - \bar{r}_2|$ not very large, the above velocity difference will depend only on the isotropic regions of turbulence $\ell_i \simeq \ell_o$: we say that the field $\bar{v}(\bar{r})$ is "locally" isotropic.

Since the field $\bar{v}(\bar{r})$ is a vector field it will be characterized by nine structure functions (instead of one) of /20

$$D_{ij}(\bar{r}) = \overline{(v_i - v_i')(v_j - v_j')}$$

where i, j = 1, 2, 3, v_i represents the component of the wind velocity at the

(1) The development will be based on the theory of Kolmogorov-Obukhov [15], [16] which today receives the larger percentage of experimental verification in relation to others which similarly describe the turbulence spectrum in the troposphere [12].

point $\bar{r}$ with respect to the axes x, y, z and v_i' its components at the point

$$\bar{r}' = \bar{r} + \bar{r}.$$

Due to the local isotropic character of the field (Landau "Fluid Mechanics", p. 124), the tensor D_{ij} depends only on $\bar{r} = r$, the unit tensor δ_{ij} and the unit vector n_i in the direction r_i.

We will have, i.e.:

$$D_{ij} = A(r)\ \delta_{ij} + \ (r)\ n_i\ n_j \qquad (1)$$

Let us now place the coordinate axes so that one of them coincides with the direction $\bar{n}$, and v_r and v_t are the projections of the velocity v_i in the direction n and perpendicular to it. So the component D_{rr} will be the mean deviation $\overline{(v_r - v_r')^2}$ of the velocity in the direction $\bar{n}$ and between the points $\bar{r}_1$ and

$$\bar{r}_1' = \bar{r} + \bar{r},$$

while

$$D_{tt} = \overline{(v_t - v_t')^2}$$

the deviation corresponding to the plane which is perpendicular to the direction $\bar{n}$ (v_t the velocity at the point $\bar{r}_1$ in a direction perpendicular to $\bar{n}$ and v_t' the value corresponding to the point

$$\bar{r}_1' = \bar{r}_1 + \bar{r}).$$

The components D_{rr} and D_{tt} are the longitudinal and "transversal" structure functions of the field $\bar{v}(\bar{r})$ respectively.

Since $n_r = 1$ and $n_t = 0$ we have from (1)

$$D_{rr} = A(r) + \ (r) \qquad (2)$$

and $D_{tt} = A(r)$.

As a result:

$$D_{ij}\ (\bar{r}) = [D_{rr}(r) - D_{tt}(r)]\ n_i \cdot n_j + D_{tt} \cdot \delta_{ij}$$

In the case where the velocity v(r) is small compared to the velocity of sound (condition which is usually fulfilled), we can consider the fluid (atmospheric air) as noncompressible, i.e.:

$\operatorname{div} \bar{v} = 0$ and then $\dfrac{\partial D_{ij}}{\partial x_i} = 0$. Substituting in (1) we get:

$$D_{tt} = \frac{1}{2r} \cdot \frac{d}{dr}\ (r^2 \cdot D_{rr}) \qquad (2)'$$

So, the tensor D_{ij} is determined only from the component D_{dt} or D_{rr}.

Assume initially that

$$L_o >> r >> \ell_o$$

(where ℓ_o the internal scale "cut-off" of the turbulence and L_o the scale of the larger - nonisotropic - "clots" of the turbulence spectrum).

From the above we see that the velocity difference between the points $\bar{r}_1$ and $\bar{r}_1' = \bar{r}_1 + r$ will depend basically on the clots with dimensions comparable to r. As we developed in paragraph 2 of the text, the only parameter characterizing these clots is the energy which is transformed into heat per unit time and mass of the atmospheric fluid, from one class of turbulence scale to the next higher.

This loss of energy we called s. We conclude then that the structure function D_{rr} (r) will be determined only from the independent variables r and s. $D_{rr}(r)$ has dimensions of velocity square. It is easy to prove that the only combination of r and s which gives the above dimensions is the quantity $(s.r)^{2/3}$ and that it is impossible to find a combination of r and s which would lead to a nondimensional quantity. This means that $D_{rr}(r)$ will have the form

$$D_{rr}(r) = C(s.r)^{2/3}$$

where C is a constant.

The above relations could be proven also by the observation that

$$\eta \; D_{rr}(r) = \overline{[v_r(r_1 + r) - v_r(r_1)]}$$

is mainly due to clots of dimension r, i.e.

$$D_{rr}(r) \sim \hat{v}_r^2$$

But as we show in paragraph 2: $v_r \sim (s.r)^{1/3}$ and then

$$D_{rr}(r) \sim (s.r)^{2/3}$$

The quantity $D_{tt}(r)$ is determined from the relation (2):

$$D_{tt}(r) = \frac{4}{3} C(s.r)^{2/3} \quad (D_{tt} = \frac{4}{3} D_{rr})$$

Assume now that $r << \ell_o$

In this case, due to the assumed (Kolmogorov) statistical balance of the region of scale ℓ_o, inside these clots the flow will be laminar and, therefore, the velocity variations between points at distance r will be smooth: this means that the difference $v_r(\bar{r}_1) - v_r(\bar{r}_1 + \bar{r})$ can be expanded in a power series of r and, due to its infinitesimal size, we will have:

$$\bar{v}_r(\bar{r}_1) - \bar{v}_r(\bar{r}_1 + \bar{r}) = \bar{a}_o\ \bar{r}$$

where $\bar{a}_o$ is a constant vector.
Therefore:

$$D_{rr}(r) = ar^2 \qquad \text{and} \qquad D_{tt}(r) = 2ar^2 = dD_{rr}$$

We will now determine the constant a:

Expanding the relation

$$D_{ij} = \overline{(v_i - v_i')\cdot(v_j - v_j')}$$

we get (having in mind the local isotropic character of the field)

$$D_{ij} = \overline{v_i v_j} + \overline{v_i' v_j'} - \overline{v_i v_j'} - \overline{v_i' v_j}$$

Similarly, due to the local isotropic character and homogenity of the field we will have

$$v_i v_j = v_i' v_j' \qquad \text{and} \qquad \overline{v_i v_j'} = \overline{v_i' v_j}$$

hence:

$$D_{ij} = 2[\overline{v_i v_j} - \overline{v_i v_j'}] \tag{3}$$

Combining now relations (1), (2), and (3) we acquire:

$$v_i v_j' = v_i v_j - a.r^2\ \delta_{ij} + \frac{1}{2} ar\delta.\ n_i n_j$$

Differentiating this relationship we get:

$$\overline{\frac{\partial v_i}{\partial x_l} \cdot \frac{\partial v_i'}{\partial x_l'}} = 15a \quad ; \quad \overline{\frac{\partial v_i}{\partial x_l} \cdot \frac{\partial v_l'}{\partial x_i'}} = 0 \tag{4}$$

In the above relations it was taken into account that due to the viscosity of the fluid, different points of it have different "internal" velocities (v_i, v_j') to which, since the fluid from the viewpoint of external turbulence /21 scale L_o is, as a whole, in motion, we must add the transfer velocity of the regions with diameter

$$\ell \sim L_o,\ v_\ell,\ v_\ell'.$$

The coordinates (x_i, x_i'), (x_ℓ, x'_ℓ) denote respectively, reference systems inside the regions of "internal" motion and the regions of large turbulence scale $\sim L$.

Because relations (4) are valid for arbitrarily small values of r we can put $x_i \to x_i'$ then:

$$\overline{\left(\frac{\partial v_i}{\partial x_l}\right)^2} = 15a \qquad \text{and} \qquad \overline{\frac{\partial v_i}{\partial x_l} \cdot \frac{\partial v_l}{\partial x_i}} = 0 \qquad (4)'$$

It is known (see e.g. Landau "Fluid Mechanics", par. 16, p. 53) that the mean dissipated energy, due to the fluid viscosity, will be equal to:

$$\varepsilon = \frac{1}{2}\left(\frac{\mu}{\varrho}\right)\overline{(grad v)^2}$$

where v is the total velocity of the fluid of external scale L_o, i.e. the sum of the velocity that corresponds to the internal relative motion of turbulence of small scale $\ell \sim \ell_o$ and the velocity corresponding to the turbulence of external scale L_o, μ is the coefficient of viscosity of the fluid and ρ its density.

I.e.:

$$\varepsilon = \frac{1}{2}\left(\frac{\mu}{\varrho}\right)\overline{\left[\frac{\partial v_i}{\partial x_l} + \frac{\partial v_l}{\partial x_i}\right]^2} =$$

$$= \left(\frac{\mu}{\varrho}\right)\overline{\left[\left(\frac{\partial v_i}{\partial x_l}\right)^2 + \frac{\partial v_i}{\partial x_l} \cdot \frac{\partial v_l}{\partial x_i}\right]} = 15\,a\left(\frac{\mu}{\varrho}\right)$$

then

$$a = \frac{1}{15}\,\frac{\varepsilon}{\left(\frac{\mu}{\varrho}\right)}$$

therefore

$$D_{rr}(r) = \frac{1}{15}\,\frac{\varepsilon}{\left(\frac{\mu}{\varrho}\right)}\,r^2, \; (r << l_o)$$

and

$$D_{tt}(r) = \frac{2}{15}\,\frac{\varepsilon}{\left(\frac{\mu}{\varrho}\right)}\,r^2 \qquad (5)$$

Therefore, we achieved the determination of the longitudinal and transversal structure functions of the field v(r) for the cases where

$$\ell_o << r << L_o:$$

$$D_{rr}(r) = C \cdot \varepsilon^{2/3} \cdot r^{2/3} \qquad (5')$$
$$D_{tt}(r) = \frac{4}{3} \cdot C \cdot \varepsilon^{2/3} \cdot r^{2/3}$$

and $\qquad r << \ell_o \qquad (6)$

The structure function $D_{rr}(r)$ is represented in fig. 3 below. For small values of r ("cut-off" region) the curve follows the parabolic law $a \cdot r^2$ up to the value $r = \ell_o$ (minimum turbulence scale) and then the law

$$C\,\varepsilon^{2/3}\,r^{2/3},$$

which is considered valid for values

$$\ell_0 << r << L_0$$

The value ℓ_0 is determined by the point of intersection of the curves

$$\frac{\varepsilon}{15\left(\frac{\mu}{\varrho}\right)} r^2 \qquad \text{and} \qquad C \cdot \varepsilon^{2/3} r^{2/3}$$

and it is equal to:

$$l_0 = \sqrt[4]{\frac{\left(15\, C \cdot \frac{\mu}{\varrho}\right)^3}{\varepsilon}}$$

Since now the distance r between the two points in the troposphere considered increases,

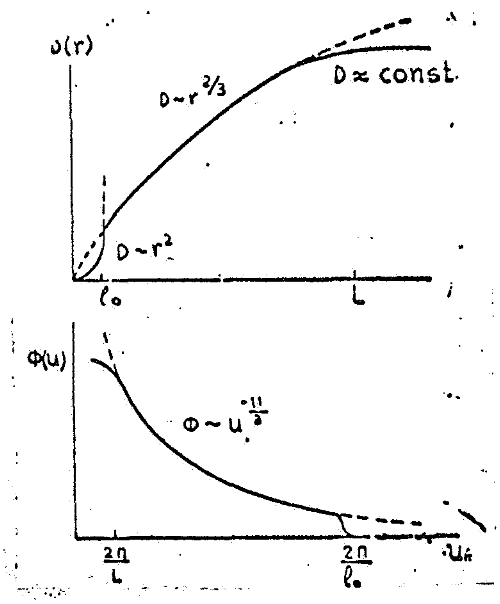

Fig. 3

the condition $r << L_0$ is no longer valid (the clots for which

$$L_0 \sim \rho > r$$

cannot be considered homogeneous and isotropic). In this case, the continuation of the curve for large values of $r \sim L_0$ ceases to follow the asymptotic law (dotted line) and "shows saturation effect" from the point of view that, for $\ell > L_0$, energy exchanges inside the fluid do not exist; it then follows a more or less linear motion absorbing, possibly, energy from the surroundings (e.g. sun energy).

4. After the calculation of the structure function $D(\bar{r})$ of the velocity field $v(\bar{r})$ of the fluid, let us now determine its energy distribution; i.e.

find the function which gives the kinetic energy of the clots at each case relative to the corresponding turbulence scales ℓ_i of them. The calculation of this function is very essential because, as we have seen, this function is the only one which enters in the determination of the power related to scattering, equivalent scattering cross section, of an electromagnetic wave propagating through the considered turbulent atmosphere.

It we designate $\Phi\ (\bar{u})$ the function of the spectral distribution to be determined, where

$$u_i = \frac{2\pi}{\ell_i}$$

(the turbulence ℓ_i plays the role in this case "of space period", hence the variable u_i represents the respective component "frequency" of the spectrum), according to the development of par. 1 of the present Appendix we will have:

$$D(\bar{r}) = 2\iiint_{-\infty}^{\infty}\left[1-\text{ουν}\,(\bar{u}\,\bar{r})\right]\Phi(\bar{u})\cdot d\bar{u}$$ /22

Under the same definition, the tensor of the structure function $D_{ij}(\bar{r})$ will give:

$$D_{ij}(\bar{r}) = 2\iiint_{-\infty}^{\infty}\left[1-\text{συν}\,(\bar{u}\,\bar{r})\right]\cdot\Phi_{ij}(\bar{u})\,d\bar{u} \tag{7}$$

where $\Phi_{ij}(u)$ will be the tensor of the energy distribution of the velocity field $\bar{v}(\bar{r})$.

The form of this last tensor $\Phi_{ij}(\bar{u})$ is determined on the basis of the same reasoning with which in the previous paragraph $D_{ij}(\bar{r})$ was expressed, i.e.:

$$D_{ij}(\bar{r})\ \text{δηλαδή}:\ \Phi_{ij}(\bar{u}) = \\ = G(u)\cdot u_i\ u_j + E(u)\,\delta_{ij} \tag{8}$$

where $G(u)$, $E(u)$ scalar functions of $\bar{u}$.

From the equation of the noncompressible fluids (for velocities $\bar{v}\ (\bar{r})$ negligible with respect to the sound velocity), it follows as in the previous paragraph:

$$\frac{\partial D_{ij}}{\partial x_i} = 0$$ and by substitution in equation (7):

$$\iiint_{-\infty}^{\infty}\text{ημ}\,(\bar{u}\cdot\bar{r})\,u_i\,\Phi_{ij}(\bar{u})\,d\bar{u} = 0 \quad \text{i.e.}\ u_i\,\Phi_{ij}(\bar{u}) = 0$$

Replacing this relation in (8) we have:

$$G(u) \cdot u^3 \cdot u_i + E(u) \cdot u_i = 0 \quad \text{or}$$

$$G(u) = -\frac{E(u)}{u^2} \qquad \text{then:}$$

$$\Phi_{ij}(\bar{u}) = (\delta_{ij} - \frac{u_i u_j}{u^2}) \cdot E(u)$$

and (7) gives:

$$D_{ij}(\bar{r}) = 2 \iiint_{-\infty}^{\infty} [1 - \sigma\upsilon\nu(\bar{u} \cdot \bar{r})] \left(\delta_{ij} - \frac{u_i \cdot u_j}{u^2}\right) \cdot E(u) \cdot d\bar{u}$$

To give the physical meaning of the function E(u), we will assume for a moment that the velocity field $\bar{v}(\bar{r})$ is isotropic and that the tensor of the correlation function of it B_{ij} exists, as well as the structure function of D_{ij}:

$$B_{ij} = \iiint_{-\infty}^{\infty} \sigma\upsilon\nu(\bar{u}\,\bar{r}) \left(\delta_{ij} - \frac{u_i u_j}{u^2}\right) E(u)\, d\bar{u}$$

in the case of isotropic conditions

$$\delta_{ij} \rightarrow \delta_{ii} = 3 \qquad \text{and} \qquad u_i u_i = u^2$$

hence:

$$B_{ii}'(\bar{r}) = \iiint_{-\infty}^{\infty} \sigma\upsilon\nu(\bar{u}\,\bar{r}) \cdot 2E(u) \cdot d\bar{u}. \qquad \Delta\iota\alpha\ r = 0$$

For r = 0 we get:

$$\frac{1}{2} \cdot \bar{v^2} = \iiint_{-\infty}^{\infty} E(u) \cdot d\bar{u}$$

Therefore, the function E(u) gives (in a three-dimensional vector space) the spectral distribution (i.e. the distribution per scale of turbulence ℓ_i) of the referenced (per unit of fluid mass) kinetic energy.

We will calculate the function E(u) in our case where the structure function of the velocity field follows the "two thirds" law which was found by the theory of Kolmogorov-Obukhov.

We have: (for the isotropic case)

$$D_{ii}(r) = 4 \cdot \iiint_{-\infty}^{\infty} \left\{1 - \sigma\upsilon\nu(\bar{u}\,\bar{r})\right\} \cdot E(u)\, d\bar{u}$$

But from (1) it follows:

$$D_{ii}(r) = D_{rr}(r) + 2D_{tt}(r).$$

Substituting

$$D_{rr}(r) = C.\ s^{2/3}\ r^{\ 2/3} \qquad \text{and} \qquad D_{tt} = \frac{4}{e}\ C.\ ^{2/3}\ r^{2/3},$$

we get:

$$\frac{11}{3}\, C.\, \varepsilon^{2/3}\, r^{2/3} = 4 \int\!\!\int\!\!\int_{-\infty}^{\infty} \left[1 - \cos(\bar{u}\ \bar{r})\right] .\ E(u).\ d\bar{u}$$

from which it follows:

$$E(u) = A.\ ^{7/8}\ u^{-11/8} \qquad (8)$$ where A is a constant equal to:

$$A = C.\ \frac{11 .\ \Gamma\left(\frac{8}{3}\right) \eta \mu \frac{\pi}{3}}{24\,\pi^2} = 0{,}06 .\ C$$

[where $\Gamma(\underline{8}/3)$ is the known gama function determined as

$$\Gamma(z) = \int_0^{\infty} e^{-t} .\ t^{z-1}\ dt \quad (Re[z] > 0)$$

with basical property:

$$\Gamma(z + 1) = z\ .\ \Gamma\ (z)$$

The function E(u) (or $\Phi(u)$) is shown in fig. 3 together with D(r). The law of variation

$$\Phi(u) \sim u^{-11/3}$$

is followed inside the region "of energy exchange"

$$\frac{2\Pi}{L_o} \ll u \ll \frac{2\Pi}{\ell_o}$$

where D(r) follows the "two thirds" law.

In the small scales for which

$$u \gtrsim \frac{2\Pi}{\ell_o}$$

$\Phi(u)$ decreases quickly (according to a yet unknown law due to the fact that, owing to the viscosity of the atmospheric region under consideration in which the propagation takes place, the turbulence scales of the order $\ell_i < \ell_o$ are expanding more or less "abruptly" into heat, the kinetic energy transferred by the larger scales.

In the large scales for which

$$u < \frac{2\Pi}{L_o}$$

$\Phi(u)$, depending exclusively on the special conditions of large scale flow, does not follow a concrete law[1] but remains almost constant ("saturation" region) due to the relative energy stability of the regions $\ell_i \sim L_o$.

The problem which arises at this point is the possibility of experimental examination of the power for the real atmosphere) of a proposed model $\Phi(u)$?[2] /23

It would be necessary then for the function $\Phi(u)$ to enter in the calculation of magnitude, easily measurable and easily distinguishable from other coexisting parameters.

In such a magnitude the received, due to scattering, power will be examined at the receiver for a radioelectric coupling occurring inside turbulent atmosphere. Whether the considered coupling occurs under line of sight contact or beyond the radioelectric horizon will have, as we will see below, as a result, some modifications on the general relations to be examined.

Let us begin with the relation, which was found in paragraph 3 of the text, between the equivalent elementary scattering cross section and the spectrum of energy distribution.

This is:

$$\sigma = \frac{64\pi^5}{\lambda^4}\,\Phi(u) = \frac{64.\pi^5}{\lambda^4}\cdot\Phi\left(\frac{4\pi}{\lambda}\,\eta\,\mu\,\frac{\Theta}{2}\right)$$

We are reminding that σ gives the received power due to scattering per unit density of incident power in the scattering volume per unit of solid angle and unit of common volume.

Let w be the transmitted power. The energy density at a point of the common volume at a distance R_o from the transmitter will be

$$\frac{\omega \quad G}{4\Pi R_o^2}$$

where G is the gain of the transmitting antenna equal to $\frac{4\Pi A}{\lambda^2}$

---

(1) Due to the same reason it is also not possible to determine the constant C but only under special assumptions [5] of ambiguous power.

(2) Here we will refer to the simplest "static" indirect method, for radioelectric couplings, which permits the determination of the spectrum in percentages of large time intervals of continuous study, and not the method by electronic refractometer [8] which permits the direct determination of $B(\bar{r})$.

where A is the equivalent surface of the antenna. The per unit of solid angle, scattered power from the volume element dV will be:

$$\frac{\omega}{4\Pi R_o^2} \cdot \frac{4\Pi A}{\lambda^2} \cdot \delta \cdot dV$$

and the received, at the position of the reception antenna (which is similar to the transmitting antenna), element of power will be:

$$dP = \frac{\omega}{4\Pi R_o} \; \frac{4\Pi A}{\lambda^2} \; \delta \cdot dV \cdot \frac{A}{R^2}$$

where R, the distance of the volume element dV from the receiving antenna.

Integrating on the entire, common scattering volume we get the total received power due to scattering

$$P = w. \frac{A^2}{\lambda^2} \int_V \frac{\delta}{R_o^2 . R^2} dV$$

It is common in the calculation for the above power to be compared with that which would be received in free space (i.e. in an isotropic and homogeneous atmosphere).

The power for free space coupling as it is known is

$$Po = w \; \frac{(4\Pi A)^2}{(\lambda^2)} \cdot \left(\frac{\lambda}{4\Pi d}\right)^2 = w \; \frac{A^2}{\lambda^2 d^2}$$

where d is the distance between transmitter and receiver.

We will accordingly have:

$$\frac{P}{P_0} = d^2 \int_V \frac{\sigma(\Theta, \lambda)}{R_0^2 \, R^2} dV = \frac{64\pi^5}{\lambda^4} \cdot d^2 \int \frac{\Phi\left(\frac{4\pi}{\lambda} \eta\mu \frac{\Theta}{2}\right)}{R_0^2 . R^2} dV \qquad (9)$$

In the case of isotropic turbulence:

$$\frac{P}{Po} = d^2 . \; \delta \int \frac{dV}{R_o^2 R^2}$$

If the coupling is of large length and beyond the horizon:

$$R_o \simeq R \simeq \frac{d}{2}$$

and then

$$\frac{P}{P_o} \quad \frac{16}{d^2}\,\delta \cdot V_\mu$$

Let us calculate the order of magnitude of the mean scattering volume $V\mu$: In a case for which the opening of the main lobes Ω of radiation of the antennas used is smaller than the angle θ

$$(\Omega = \frac{\lambda^2}{A} < \theta$$

simple geometric considerations give

$$V_\mu \sim \frac{d^3\Omega^3}{8\theta}$$

Therefore:

$$\frac{P}{P_o} \sim 2\delta \frac{d}{\theta}\,\Omega^3$$

But

$$\frac{d}{\quad} \quad R_f$$

where R_f is the equivalent transformed radius of the Earth, determined from the equivalent mean slope of the refraction coefficient inside the coupling region.

Correspondingly:

$$\frac{P}{P_o} \simeq R_f . \delta \cdot \Omega^3 \sim R_f \cdot \frac{\Omega^3}{\lambda^4} \cdot \phi\,(u) \qquad (10)$$

In the case now of line of sight contact, we must consider the scattered power in all directions:

$$\Sigma = \int_{4\Pi} \delta . d\Omega$$

The scattered power for length d (of the coupling length) will be

$$P = P_o \cdot e^{-\Sigma . d}$$

where P_o the power in free space. Therefore, we will have at the receiver, power arriving due to scattering:

$$P = P_o(1 - e^{-\Sigma d}) = P_o\,[\Sigma . d + \frac{1}{2}(\Sigma . d)^2 + \ldots .]$$

For $\Sigma . d \ll 1$
(Born Criterion - scattering "weak")

$$P_d \sim P_o . \Sigma . d$$

or more accurately, taking into account the reflected beam on the Earth's surface (coming from the radiation pattern which is located below the line between transmitter and receiver),

$$P_d \sim P_o . \Sigma . d . q$$

where q is the reflection coefficient of the surface.

[By assuming, for example, turbulence spectrum of Gauss' form it follows

$$\sigma = \frac{\sqrt{\pi}}{2\lambda} \left(\frac{2\pi l}{\lambda}\right)^4 \overline{\Delta n^2} \, exp \left[- \left(\frac{2\pi l}{\lambda}\right)^2 \eta \mu^2 \frac{\Theta}{2} \right]$$ /24

and $\Sigma \cong \frac{8\pi^4}{\lambda^4} \overline{\Delta n^2} . l$ (for $2\pi l/\lambda >> 1$))

where ℓ is the mean scale of turbulence and $\Delta\eta^2$, the mean deviation of the refraction index. Hence:

$$\frac{P_d}{P_o} \sim \frac{8\pi^4 . \overline{\Delta n^2} . l}{\lambda^4} d.$$

We observe, i.e. that the scattered component received decreases as d^{-1}, while the direct component in free space, as d^{-2} beyond the same point (or above a certain region) direct and scattered components combine at the receiver with comparable magnitudes.]

Assuming now the spectrum of Kolmogorov-Obukhov

$$\Phi(u) \sim \bar{l}^{-4/3} . \bar{u}^{-11/3} \sim \bar{l}^{-4/3} \left(\frac{1}{\lambda} \eta \mu \frac{\Theta}{2}\right)^{-11/3} \sim \bar{l}^{-4/3} . \left(\frac{\Theta}{\lambda}\right)^{-11/3}$$

and substituting in (10) we get:

$$\frac{P}{P_o} \sim R_f \; \Omega^2 . \bar{\lambda}^{-1/3} \; \bar{\Theta}^{-11/3} \; \bar{l}^{-4/3} \sim \Omega^2 \bar{\lambda}^{-1/3} \bar{d}^{-11/3} \bar{l}^{-4/3} R_f^{14/3} (11) \qquad (11)$$

The relation (11) shows a dependence

$$\bar{\lambda}^{-1/8}$$

between the received power and the wave length, as long as we have assumed the above spectral distribution.

The experimental verifications though [14] (in networks both beyond the horizon and line of sight contact) show that there is not only one dependence λ^{-K} but, depending on the geometry of coupling and the season and the time of the experiment, the parameter K fluctuates approximately between $-3<K<3$ for couplings beyond the horizon, and decreases monotonically [13] for line of sight couplings.

A presentation of this differentiation together with related experimental data of the above Greek atmospheric area is given in one of our recent studies [13]. One could say accordingly that a relation of the form

$$\Phi(u) \sim u^{-x} \qquad (12)$$

is not valid at all times for the same x, which means that, because in the same space the only parameter which changes the received power in a given radioelectric coupling in time is, in the last analysis, the turbulence scale ℓ_i, the exponent x must be different for different spectrum regions

$$\ell_o << \ell_i << L_o$$

Substituting in fact (12) in (9) and assuming that for the same coupling we use simultaneously two wave lengths λ_1 and λ_2 with proper antennas so that, in both cases, the common scattering volume is the same, by comparing simultaneous mean values of reception we obtain:

$$\frac{P(\lambda_1)}{P(\lambda_2)} \sim \left(\frac{\lambda 1}{\lambda 2}\right)^{x-4} \qquad (13)$$

The simultaneous experimental verifications give x-4 between 2 and -2, i.e. $+2 \leq x \leq 6$ meaning spectrum $\Phi(u)$ inside the region of energy exchanges

$$\Phi(u) \sim u^{-2} \quad \text{until} \quad \Phi(u) \sim u^{-6}$$

The above reasoning though is not absolutely correct because it is based on the assumption that the only propagation mechanism of the electromagnetic radiation through the troposphere is the one of refraction (of scattering) by the turbulent regions ℓ_i and, furthermore, that in this last case the only mechanism of transforming kinetic energy into heat is the one of the fluid viscosity and that the turbulence strength is constant, leading always to a scale of cut-off ℓ_o of the same order of magnitude (mm).

It has been known for some time though (Friis, Crawford, Hogg. Voge, Du Castel, Misme) that in the troposphere, the tracking (by radio sounding and electronic refractometers) "of foils" regions thermodynamically stable of large horizontal dimensions (a few km) compared to the vertical dimensions (a few m) which are characterized by sharp vertical slope of their refraction index relative to the mean slope of the surrounding (turbulent) space, is possible. Each of the "foils" creates a partial reflection of the incident wave.

For a "foil" of infinitesimal thickness corresponding to a small discontinuity dn of the refraction index, the reflection coefficient will be

$$dr = \frac{dn}{2a^2}$$

where a is the incident angle ($a \sim \frac{z}{d}$ of the order of a few mrad). For a "foil" situated at a height z of small thickness h, the reflection coefficient r is the Fourier transformation of the change of slope

$$\frac{dn(z)}{dz} = g(z)$$

of the refraction index inside this "foil":

$$r = \int_{z}^{z+h} \frac{g(z)}{2a^2} e^{-j2kz} \cdot dz \qquad \text{where} \qquad k = \frac{2\pi}{\lambda} \eta\mu a \smile \frac{2\pi}{\lambda} a$$

(2kz: coefficient which takes into account the phase difference due to reflection).

Obviously, the value of r depends on the form of stratification g(z) inside the "foil".

If, for example, we assume linear variation leading to a difference δg of the slope of the reflection index through the thickness h of the "foil", we will have:

$$r = \frac{\delta g}{8\pi a^3} \lambda . e^{-j\frac{4\pi}{\lambda} a(z+h)}$$

This relation is valid as long as the horizontal dimensions of the "foil" are assumed to be infinitely large and plane. Given though, the experimental verifications leading to finite values, L_o of horizontal dimensions and, in the existence of fluctuations of their separating surfaces, the coefficient r must be multiplied by a factor which will take into account the relation of the dimensions L_o to the corresponding dimensions of the first Fresnel belt, which constitutes the cross section between the first Fresnel elipsoid with focal points, the transmitter and the receiver and the reflecting surface of the "foil", as well as, in fact, the separating surfaces not being smooth in which cases, the Rayleigh criterion is not fulfilled. If d is the distance between transmitter and receiver, the dimensions of the above Fresnel belt (ellipse) will be

$$\sqrt{\lambda d}$$

(transversal) and

$$\frac{\sqrt{\lambda d}}{a}$$

(longitudinal), (where at the incident angle in rad).

For wave length λ of the microwave region we usually have /25

$$\sqrt{\lambda d} < L_o < \sqrt{\frac{\lambda d}{a}}$$

hence the reflection coefficient r becomes:

$$|r_o| = r \cdot \frac{L_o}{\sqrt{\lambda d}} a$$

So, the contribution of each "foil" will be manifested by one "elementary referenced reflected power"

$$\sigma_o \sim |r_o|^2 = \left(\frac{1}{8\pi}\delta g \cdot L_o \cdot \lambda^{1/2} \cdot a^{-2} d^{-1/2}\right)^2 \sim \overline{\delta g^2} \cdot L_0^2 \cdot a^{-4} \cdot d^{-1} \cdot \lambda$$

of phase difference

$$\phi = \frac{8\Pi}{\lambda} \cdot h.a$$

relative to the incident power on the "foil". The total contribution of power, due to the mechanism of partial reflections of the atmospheric "foils" of different thickness, horizontal dimensions, and altitudes, will result from addition of the individual elements of power $\sigma_o(h)$ (taking naturally into account the phase differences at each case).

Therefore, the received power, relative to the corresponding power in free space, will be:

$$\frac{P}{P_o} = \sum_h \sigma_o(h) \quad * \qquad \text{or} \qquad \boxed{\frac{P}{P_o} \sim d^{-5} \cdot \lambda \cdot L_0^2} \qquad (14)$$

---

*More accurately, it will be: $P_r / P_o = \sum \frac{d^2}{4\pi d_1^2 d_2^2} \cdot \sigma_{o\lambda}$

where d_1, d_2 are the distances from the transmitter and the receiver to the point of geometric reflection on the "foil" under consideration and:

$$\sigma_{o\lambda} = \frac{4\pi}{\lambda^2} \left| \iint_s e^{-jk.\delta l} \cdot r \cdot ds \right|^2 \qquad \text{where} \qquad k = \frac{2\pi}{\lambda}$$

and $\delta\ell$ is the difference in the path $(d_1 + d_2) - (d_1' + d_2')$ between the point of geometrical reflection on the "foil" and the center of the next elementary surface ds of the "foil". The above expression must be taken into account when we are considering "foil" of large horizontal dimensions and of wavy separating surfaces.

From relations (11) and (14) it is obvious that the two mechanisms, scattered and partial reflections, lead in comparable results to the order of magnitude of influence of the main coupling parameters, i.e. the distance and the frequency. In the case of the mechanism of propagation through regions of turbulent atmosphere, the nonisotropic character of the medium brings only slight modifications in the facts resulting from the assumption of isotropic turbulence, due to the fact that the vertical and horizontal dimensions of the clots under consideration are similar.

In the case of the mechanism by partial reflections, the horizontal and vertical dimensions of the "foils" considered differ by many orders of magnitude and naturally play completely different roles.

In the case of scattering, the limits of the turbulence scale inside which energy transfer occurs, (i.e. turbulence) $\ell_1 \sim \ell_o$ and $\ell_2 < L_o$ are of much different order of magnitude ($\ell_1 \sim 1$ mm $\ell_2 \sim 100$ m). As a result (11) indicates that the magnitudes $\ell_1 \ldots\ldots\ldots \ell_2$, for a given coupling length d or θ, are comparable to the wave length λ. We conclude then that the ratio $\frac{P}{P_o}$ must depend on the same law (the same x)

$$\frac{P}{P_o} \sim$$

for a very large frequency region, which contradicts the experimental verifications [13].

In the case of partial reflections on the contrary, the corresponding limits are very much different: Primarily, the range of the thickness h of the different "foils" (which varies from a few meters to a few meter decades) must be compared to the phase difference

$$\frac{4\Pi}{\lambda} \cdot a \cdot h$$

i.e. to the magnitude $\frac{\lambda}{a}$.

On the other hand, the horizontal dimensions L_o of the "foils" (of the order of a few km) must be compared to the axis of Fresnel ellipse

$$\sqrt{\lambda . d} \qquad \text{and} \qquad \sqrt{\lambda . d / a}.$$

Therefore, the dependence of the received power on the wave length, in this case, must be much more critical, a fact that justifies the spread of the parameter x in the formula

$$\frac{P(\lambda_1)}{P(\lambda_2)} \mp \frac{(\lambda_1)^x}{(\lambda_2)}$$

without necessarily this spread being caused by a corresponding change in the form of the turbulence spectrum.

Finally it is worth noting the fact that the experimental (from the morphology of the variation patterns of the received electromagnetic field in each case) separation of each of the above propagation mechanisms is rather clear and simple: In the case of scattered signal due to turbulence, the statistical nature of the propagation medium is expressed twice: a) by introduction of the energy spectrum $\Phi(u)$ (or the correlation function $B(r)$ of the refraction index fluctuations) directly in the same elementary (referenced) power of reception $\sigma d(\theta,\lambda)$ (the equivalent scattering cross section) and b) the integral which comes after $\sigma d(\theta,\lambda)$ for the calculation of $P_{\text{reception}}$, taking into account a second statistical distribution of $\sigma d(\theta,\lambda)$ in space (received signal made up by a very large number of components of random phase).

In the case of a signal from partial reflections, the reflection coefficient of each atmospheric "foil" contains its own phase and only from the addition

$$\sum_h \delta_o(h)$$

of the respective elements of reflected power the statistical character of the phenomenon appears (once). Reception signal made up by a very small number of components of random phase.

According to the above developments though, the dependence of the received power on the wave length, through the troposphere, is not expected, theoretically, to have range greater than

$$\lambda^{-1/2} \quad \lambda^{1}$$

This disagreement with the above developed, experimental verifications for beyond horizon couplings

$$(-\lambda^{-2}\ldots\ldots\lambda^{2})$$

could be removed, if the scale of turbulence of cut-off /26

$$l_0 \doteq \sqrt[4]{\frac{(\mu/\varrho)^3}{\sigma}}$$

would remain for a sufficient percentage of the observation time in the meter size or the 100 m size region instead of the mm size region. In this case the curve $\Phi(u)$ would be separated in two main branches from which the second (which refers to the values $\ell_i \sim \ell_o$) would be decreasing much faster than the first, and for a scale interval $\Delta\ell$ comparable to that of the first, which would justify the experimental results. Large values of ℓ_o correspond to very small values of expanded power per scale of turbulence due to the viscosity σ. Very small values σ, though, denote either a corresponding weak turbulence or a faster reduction in the density of the kinetic energy, towards the smaller values ℓ_i.

The last assumption results in a progressive "absorbtion" of kinetic energy by some mechanism different than the one of the viscous forces. What type of mechanism is this? According to an assumption [Bolgiano 1959] which is verified very satisfactorily by experiment, the retarding forces on moving clots inside atmospheric, thermodynamically stable "foils" [9], which force these clots in a damped oscillation that brings them in the original state of equalibrium, transform part of their kinetic energy into potential energy, on a scale of many orders of magnitude higher than σ. The above retarding forces increase with the magnitude of the corresponding clots; the above mechanism, therefore, (which produces an increase in slope of the first left member of $\Phi(u)$) shows basically in the high nonisotropic (due to "layerlike" nature of the surrounding) scale of turbulence.

In conclusion: the form of $\Phi(u)$ is subdivided into four regions: the region of "inertia" $\ell_i \sim \ell_o$, the region "of retardation" in which the kinetic energy of the corresponding scales is absorbed by the viscous forces of the thermodynamically stable fluid being transformed in potential energy, the region in which the energy absorbtion occurs, in friction energy quantities σ and finally the "cut-off region" $\ell_i \sim \ell_o$.

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